1 Fundamentals

1.1 Real Numbers

To start, we'll review the numbers that make up the real number system.

1. **Natural numbers** (denoted $\mathbb{N}$) - these are just positive whole numbers:
$$1, 2, 3, \ldots$$
2. **Integers** (denoted $\mathbb{Z}$) - all natural numbers together with their negatives and 0:
$$\ldots, -2, -1, 0, 1, 2, \ldots$$
3. **Rational Numbers** (denoted $\mathbb{Q}$) - all ratios of integers:
$$\frac{1}{2}, \frac{107}{125}, \frac{m}{n} \quad \text{where } m, n \text{ are integers and } n \neq 0 \text{ (since } n = 0 \text{ is not defined)}$$
4. **Irrational numbers** - numbers that cannot be expressed as a ratio of integers:
$$\sqrt{2}, \sqrt{5}, \pi$$

The set of all **real numbers** is denoted $\mathbb{R}$ and consists of all the numbers in 1-4 above. Real numbers have some great arithmetic properties.

<u>Properties of Real Numbers</u>

Let x, y and z be real numbers.

Property	**Example**
1. $x + y = y + x$	$7+3 = 3+7$
2. $xy = yx$	$3 \cdot 5 = 5 \cdot 3$
3. $(x + y) + z = x + (y + z)$	$(2 + 4) + 7 = 2 + (4 + 7)$
4. $(xy)z = x(yz)$	$(3 \cdot 7) \cdot 5 = 3 \cdot (7 \cdot 5)$
5. $x(y + z) = xy + xz$	$2(3 + 5) = 2 \cdot 3 + 2 \cdot 5$
6. $(y + z)x = xy + xz$	$(3 + 5)2 = 2 \cdot 3 + 2 \cdot 5$
7. $x + 0 = x$	$5 + 0 = 5$
8. $x + (-x) = 0$ and $-x = (-1)x$	$5 + (-5) = 0$ and $-5 = (-1)5$
9. $x = y$ if and only if $x + z = y + z$	$5 = 5$ if and only if $5 + 2 = 5 + 2$
10. $x \cdot 1 = x$ and $x \cdot 0 = 0$	$6(1) = 6$ and $6(0) = 0$
11. If $x \neq 0$, then $x \cdot \frac{1}{x} = 1$	$6 \cdot \frac{1}{6} = 1$

Continued:

12. $x = y$ if and only if $xz = yz$ with $z \neq 0$	$x = y$ if and only if $6x = 6y$
13. If $xy = 0$, then $x = 0$ or $y = 0$	If $6x = 0$, then $x = 0$
14. $\frac{x}{z} + \frac{y}{z} = \frac{x+y}{z}$ with $z \neq 0$	$\frac{1}{2} + \frac{3}{2} = \frac{1+3}{2} = \frac{4}{2} = 2$
15. $\frac{x}{z} + \frac{y}{w} = \frac{xw+yz}{zw}$ with $z, w \neq 0$	$\frac{1}{2} + \frac{1}{3} = \frac{1\cdot 3+1\cdot 2}{2\cdot 3} = \frac{5}{6}$
16. $\frac{x}{z} - \frac{y}{w} = \frac{xw-yz}{zw}$ with $z, w \neq 0$	$\frac{1}{2} - \frac{1}{3} = \frac{1\cdot 3-1\cdot 2}{2\cdot 3} = \frac{1}{6}$
17. $\frac{x}{z} \cdot \frac{y}{w} = \frac{xy}{zw}$ with $z, w \neq 0$	$\frac{1}{2} \cdot \frac{5}{3} = \frac{5}{6}$
18. $\frac{\frac{x}{z}}{\frac{y}{w}} = \frac{x}{z} \cdot \frac{w}{y}$ with $z, w, y \neq 0$	$\frac{\frac{1}{2}}{\frac{5}{3}} = \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}$
19. $\frac{zx}{zy} = \frac{x}{y}$ with $z, y \neq 0$	$\frac{6x}{6y} = \frac{x}{y}$

Sets and Intervals

Definition 1.1 *A* **set** *is a collection of objects, and these objects are called* **elements** *of the set.*

Example: $A = \{1, 2, 3, 4\}, B = \{\frac{1}{2}, \pi, 0, 5\}$, and $C = \{x_1, x_2, x_3, \dots\}$ are all sets.

If S is a set, then the notation $x \in S$ means that "x" is an element of S, and $y \notin S$ means that "y" is not an element of S. For instance, in the above example $2 \in A$ and $2 \notin B$.

Definition 1.2 *If S and T are sets, then their* **union** *(denoted $S \cup T$) is the set that consists of all elements that are in S or T (or in both). The* **intersection** *of S and T (denoted $S \cap T$) consists of all elements that are common to both S and T. The* **empty set***, denoted $\emptyset$, is the set that contains no elements.*

Example: If $S = \{1, 2, 3, 4, 5\}$, $T = \{4, 5, 6, 7\}$, and $V = \{6, 7, 8, 9\}$, then

$$\begin{aligned} S \cup T &= \{1, 2, 3, 4, 5, 6, 7\} \\ T \cup V &= \{4, 5, 6, 7, 8, 9\} \\ S \cap T &= \{4, 5\} \\ S \cap V &= \emptyset \end{aligned}$$

There are two more symbols we use to relate sets. If A and B are sets, then the set $A - B$ is the set of all elements in A that are not in B. For example,

$$\{2, 4, 6, 8\} - \{2, 4\} = \{6, 8\}$$

$$\mathbb{R} - [0, \infty) = (-\infty, 0)$$

If we have two sets A and B such that every element of B is also in A, then we say that B is **contained** in A and write $B \subset A$. We also say that B is a **subset** of A. For example

$$\{1, 2\} \subset \mathbb{Z}$$

$$(1,2) \subset (0,4)$$

We can also write a set using another notation. If we write

$$\{n \mid n \text{ is an even natural number}\}$$

it is read as " the set of all n such that n is an even natural number". The vertical line stands for "such that". Thus

$$\{n \mid n \text{ is an even natural number}\} = \{2, 4, 6, \ldots\}$$

One very important example of a set is an interval. There are a few different types, so let's define them.

Definition 1.3 *An* **open interval** *from a to b, where $a < b$, consists of all real numbers between a and b and is denoted (a, b). A* **closed interval** *from a to b is the same as the open interval except that it includes the endpoints and is denoted $[a, b]$.*

Graphs:

Open interval:

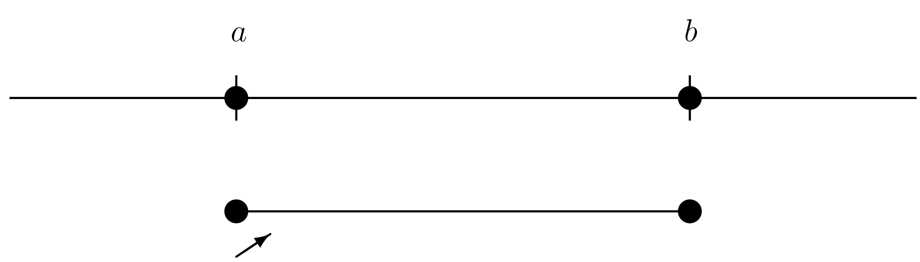
$$(a, b) = \{x \mid a < x < b\}$$

the open inverval (a, b)

The open circle at the ends of the interval indicates that the endpoint is not included.

Closed interval:

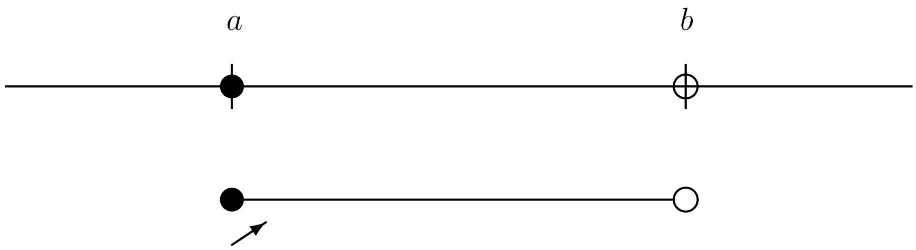
$$[a, b] = \{x \mid a \leq x \leq b\}$$



the closed inverval $[a, b]$

There is a mixed type of interval called a **half-open** (or **half-closed**!) interval. These intervals include one endpoint and not the other. For example:

$$[a, b) = \{x \mid a \leq x < b\}$$



the half-open interval $[a, b)$

The last case is where one of the endpoints is ∞. These intervals are denoted thusly:

$$(a, \infty) = \{x \mid a < x\}$$

$$[a, \infty) = \{x \mid a \leq x\}$$

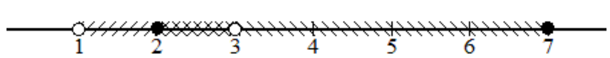
$$(-\infty, b) = \{x \mid x < b\}$$

$$(-\infty, b] = \{x \mid x \leq b\}$$

The interval $(-\infty, \infty)$ is, of course, all real numbers $\mathbb{R}$.

Examples:

1. $[-1, 2] = \{x \mid -1 \leq x \leq 2\}$
2. Consider the intersection of two intervals $(1, 3) \cap [2, 7]$. Remember that the intersection of two sets is the elements in common between the two sets. In the case of intervals, the intersection of two intervals is therefore where they overlap. In the below image, the shading indicates $[2, 7]$, the shading indicates $(1, 3)$, and indicates the overlap. Thus, the intersection is $[2, 3)$.



Absolute Value and Distance

Definition 1.4 *If a is a real number, then the* ***absolute value*** *of a is definied as*

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a \leq 0 \end{cases}$$

Examples:

1. $|3| = 3$
2. $|-3| = 3$
3. $|0| = 0$

The absolute value of a number is like the "positive part" or "magnitude" of the number.

Properties of Absolute value

Property	**Example**
1. $\|a\| \geq 0$	$\|-3\| = 3 \geq 0$
2. $\|a\| = \|-a\|$	$\|5\| = \|-5\| = 5$
3. $\|ab\| = \|a\|\|b\|$	$\|-2 \cdot 5\| = \|-2\|\|5\| = 10$
4. $\left\|\frac{a}{b}\right\| = \frac{\|a\|}{\|b\|}$	$\left\|\frac{12}{-3}\right\| = \frac{\|12\|}{\|-3\|} = \frac{12}{3} = 4$

Definition 1.5 *The* **distance** *between two real numbers a and b is $d(a,b) = |a-b| = |b-a|$.*

Example: The distance between -8 and 2 is

$$d(2,-8) = |-8-2| = |-10| = 10$$

1.2 Exponents and Radicals

Definition 1.6 *If x is any real number and n is a positive integer, then*

$$x^1 = x$$

$$x^2 = x \cdot x$$

$$x^3 = x \cdot x \cdot x$$

$$\cdots$$

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ times}}$$

Here, the number x is called the **base** *and the number n is called the* **exponent**. *Additionally, if $x \neq 0$, we have*

$$x^0 = 1$$

$$x^{-1} = \frac{1}{x}$$

$$x^{-2} = \frac{1}{x^2}$$

$$\cdots$$

$$x^{-n} = \frac{1}{x^n}$$

Examples:

1. $7^2 = 7 \cdot 7 = 49$
2. $(\frac{1}{2})^5 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$

Properties of Exponents

Let $a, b \neq 0$ be real numbers and let $m, n \in \mathbb{Z}$. Then

Property	**Example**
1. $a^n \cdot a^m = a^{n+m}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^5}{3^2} = 3^{5-2} = 3^3 = 27$
3. $(a^m)^n = a^{mn}$	$(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$
4. $(ab)^n = a^n b^n$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2 = 9 \cdot 16 = 144$
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$
6. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$
7. $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$	$\frac{3^{-2}}{4^{-5}} = \frac{4^5}{3^2}$

Examples: Simplify the following expressions:

1. $\dfrac{6st^{-4}}{2s^{-2}t^2}$

   solution:

$$\frac{6st^{-4}}{2s^{-2}t^2} = \frac{6ss^2t^{-4}}{2t^2} = \frac{6ss^2}{2t^2t^4} = \frac{6s^3}{2t^6} = \frac{3s^3}{t^6}$$

2. $\left(\dfrac{y}{3z^3}\right)^{-2}$

   solution:

$$\left(\frac{y}{3z^3}\right)^{-2} = \left(\frac{3z^3}{y}\right)^2 = \frac{3^2(z^3)^2}{y^2} = \frac{9z^6}{y^2}$$

Radicals

Definition 1.7 *Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, the* **nth root of x** *is defined as follows:*

$$\sqrt[n]{x} = y \text{ means } y^n = x$$

where $x, y \geq 0$. If n is not written, $n = 2$ ie $\sqrt[2]{x} = \sqrt{x}$.

Example: $\sqrt{4} = \pm 2$ since $(-2)^2 = 4$ and $2^2 = 4$.

In general, if $x > 0$ and n is an even number, there are two nth roots of x, say y and $-y$, since $y^n = (-y)^n = x$.

If $x > 0$ and n is even, there is no nth root of x (this doesn't depend on the sign of x or evenness/oddness of n).

Examples:

1. $\sqrt{-8}$ does not exist since $x = -8 < 0$ and $n = 2$ is even.
2. $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$. Further, since $n = 3$ is odd there is only one root.
3. $\sqrt{4^2} = \sqrt{16} = 4$, but notice that $(-4)^2 = 16$, thus $\sqrt{4^2} = \sqrt{16} = \pm 4$.

Note: For us, when there are two possibilities we will always choose the positive one. So $\sqrt{x^2} = |x|$.

Properties of Radicals

Let a, b be real numbers and let $m, n \in \mathbb{Z}$. Then

Property	**Example**
1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8} \cdot \sqrt[3]{27} = (-2)(9) = -18$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{4}{3}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$
4. $\sqrt[n]{a^n} = a$ if n is odd	$\sqrt[3]{(-5)^3} = -5$
5. $\sqrt[n]{a^n} = \|a\|$ if n is even	$\sqrt[4]{(-3)^4} = \|-3\| = 3$

WARNING: Note that, in general,

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

For instance, $\sqrt{16+9} = \sqrt{25} = 5$, but $\sqrt{16} + \sqrt{9} = 3 + 4 = 7 \neq 5$.

Now with our knowledge of integer exponents and roots we can define exponents for any rational number.

Definition 1.8 *Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. We define a* **rational exponent** *as*

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m \text{ or } a^{m/n} = \sqrt[n]{a^m}$$

If n is even, this is only defined if $a \geq 0$.

Note that the properties of exponents also hold for rational exponents.

Examples:

1. $a^{1/3}a^{7/3} = a^{1/3+7/3} = a^{8/3}$

2. $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3 \left(\frac{y^4}{x^{-1/2}}\right) = \frac{2^3(x^{3/4})^3}{(y^{1/3})^3}(y^4x^{1/2}) = \frac{2^3x^{9/4}}{y}(y^4x^{1/2}) = 8x^{11/4}y^3$

3. $\sqrt{x\sqrt{x}} = \left(x \cdot x^{1/2}\right)^{1/2} = (x^{3/2})^{1/2} = x^{3/4}$

Rationalizing the Denominator

For reasons beyond my realm of knowledge it is preferable to show algebraic expressions with no radicals in the denominator. If you have an expression with a radical in the denominator, find something to multiply the bottom by that will get rid of any roots, and then multiply the top and bottom by that thing.

Examples:

1. $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$. Note that this expression is only valid if $a > 0$.
2. $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

1.3 Algebraic Expressions

Definition 1.9 *A* **polynomial** *in the variable x is an expression of the form*

$$a_nx^n + a_{n-1}x^{n-1} + \ldots a_1x^1 + a_0$$

where the a's are real numbers and n is a natural number. If $a_n \neq 0$, then we say that the polynomial has **degree** *n.*

Examples:

1. $P(x) = x^3 + 3x + 2$ is a polynomial with degree 3.
2. $Q(x) = 1$ is a polynomial with degree 0.
3. $F(x) = x - \sqrt{2}$ is a polynomial with degree 1.

Definition 1.10 *An* **algebraic expression** *is defined as the combination of polynomials by addition, subtraction, multiplication, division, powers, and roots.*

Examples:

1. $2x^2 + 3x + 4$
2. $\sqrt{x} + 1$

3. $\dfrac{y-27}{y^2+4}$
4. $\sqrt{\dfrac{x-\sqrt{x}}{x^2+1}}$
5. $\sqrt[3]{x}$

Adding and Subtracting Polynomials

To add and subtract polynomials, group terms with like powers of x together.

Examples:

1. Find the sum $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$.

   solution:

   $$\begin{aligned}(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x) &= (x^3 + x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4 \\ &= 2x^3 - x^2 - 5x + 4.\end{aligned}$$

2. Find the difference $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$.

   solution:

   $$\begin{aligned}(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x) &= (x^3 - x^3) + (-6x^2 - 5x^2) + (2x - (-7x)) + 4 \\ &= 0x^3 - 11x^2 + 9x + 4 \\ &= -11x^2 + 9x + 4.\end{aligned}$$

Multiplying Algebraic Expressions

Examples:

1. Find the product $(2x + 1)(3x - 5)$.

   solution: To multiply two polynomials together, you take each of the terms in the first one and multiply by each of the terms of the second one and add them. So in our case, we would take $2x \cdot 3x$, and then add $2x \cdot (-5)$, and do the same for 1. So we end up with

   $$(2x+1)(3x-5) = 2x \cdot 3x + 2x \cdot -5 + 1 \cdot 3x + 1(-5) = 6x^2 - 10x + 3x - 5 = 6x^2 - 7x - 5$$

   A teacher of mine called this process "shooting the frogs into the buckets" where the first terms were "frogs" (here $2x$ and 1) which were shot into the "buckets" (the second terms, here $3x$ and -5). Note that I do not condone shooting frogs into anything as that is animal cruelty.

2. $(1 + \sqrt{x})(2 - 3\sqrt{x}) = 2 - 3\sqrt{2} + \sqrt{x} \cdot 2 - \sqrt{x} \cdot 3\sqrt{x} = 2 - \sqrt{x} - 3x$

Factoring

As above, if we are given $(x-2)(x+2)$, we can multiply it out to get

$$(x-2)(x+2) = x^2 - 4$$

Conversely, sometimes we are given an expression like $x^2 - 4$ and asked to find out if there are two smaller polynomials which multiply together to give it as a result. This is called **factoring**. Thus

$$\xrightarrow{\text{expanding}}$$

$$(x-2)(x+2) = x^2 - 4$$

$$\xleftarrow[\text{factoring}]{}$$

Another common example of factoring is finding a term common to all the terms in an expression and "bringing it out front".

Examples:

1. $3x^2 - 6x = 3x(x-2)$
2. $8x^4y^2 + 6x^3y^3 - 2xy^4 = 2xy^2(4x^3 + 3x^2y + y^2)$

Factoring Quadratics

Factoring polynomials of degree 2 (i.e. quadratics) comes up frequently. In general, if we want to factor something like $x^2 + bx + c$, we want to find two numbers p and q that add up to b and that multiply together to give c. Then we will have $x^2 + bx + c = (x+p)(x+q)$.

Example: Suppose we want to factor $x^2 - 5x + 6$ into two smaller polynomials. Then we need to find two numbers that multiply together to give 6 and that add together to give -5. After a bit of guessing, the only numbers that do this are -2 and -3. Thus we have

$$x^2 - 5x + 6 = (x-2)(x-3)$$

Multiplying the right hand side out shows that the two are indeed equal.

Factoring with Radicals

When factoring expressions involving rational exponents, factor out the lowest power present, even if it is negative.

Example: Suppose we want to factor $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$. The lowest power present is $x^{-1/2}$, so we bring it outside along with the common factor of 3:

$$\begin{aligned} 3x^{3/2} - 9x^{1/2} + 6x^{-1/2} &= 3x^{-1/2}(x^2 - 3x + 2) \\ &= 3x^{-1/2}(x-2)(x-1) \end{aligned}$$

The last step is because -1 and -2 add up to -3 and multiply to give 2.

Factoring by Grouping

Higher-degree polynomials are a bit trickier to factor, but some can be done by some clever groupings. For example, consider $x^3 + x^2 + 4x + 4$. The first two terms share a factor of x^2 while the last two share a factor of 4. Hence

$$\begin{aligned} x^3 + x^2 + 4x + 4 &= (x^3 + x^2) + (4x + 4) \\ &= x^2(x + 1) + 4(x + 1) \end{aligned}$$

Now the first group and second group have a common factor of $(x - 1)$, so we can factor it out to get

$$x^3 + x^2 + 4x + 4 = (x + 1)(x^2 + 4)$$

Some Generally Useful Formulas

Formula	**Example**
1. $(a-b)(a+b) = a^2 - b^2$	$(2x-5)(2x+5) = 4x^2 + 10x - 10x + 25 = 4x^2 + 25$
2. $(a+b)^2 = a^2 + 2ab + b^2$	$(x+5)(x+5) = x^2 + 5x + 5x + 25 = x^2 + 10x + 25$
3. $(a-b)^2 = a^2 - 2ab + b^2$	$(x-5)(x-5) = x^2 - 5x - 5x + 25 = x^2 - 10x + 25$
4. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	$(x+5)^3 = x^3 + 5x^2 + 25x + 125$
5. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$	$(x-5)^3 = x^3 - 5x^2 + 25x - 125$
6. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$	$x^3 + 8 = x^3 + 2^3 = (x+2)(x^2 - 2x + 4)$
7. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$	$27x^3 - 1 = (3x)^3 - 1^3 = (x-1)(9x^2 + 3x + 1)$

1.4 Rational Expressions

Definition 1.11 *A quotient of two algebraic expressions is called a* **fractional expression***. A* **rational expression** *is a fractional expression where both the numerator and denominator are polynomials.*

For example, $\dfrac{\sqrt{x}+1}{x^2+4x}$ is a fractional expression while $\dfrac{x-2}{x^2-5x+6}$ is a rational expression.

The Domain of an Algebraic Expression

Definition 1.12 *The* **domain** *of an algebraic expression is the set of values that the variable is allowed to take.*

Examples:

1. The expression $\dfrac{1}{x}$ has domain $\{x \in \mathbb{R} \mid x \neq 0\}$, that is, the set of all numbers except 0. This is because you cannot divide by 0.
2. The expression $\sqrt{x}$ has domain $\{x \in \mathbb{R} |\ x \geq 0\}$, that is, the set of all non-negative numbers. This is because you cannot take the square root of a negative number. In interval notation, the domain of this expression is $[0, \infty)$.
3. The expression $\dfrac{1}{\sqrt{x}}$ has domain $\{x \in \mathbb{R} \mid x > 0\}$, that is, the set of all non-negative numbers except 0. This is because you cannot take the square root of a negative number nor can you divide by 0. In interval notation, the domain of this expression is $(0, \infty)$.
4. Find the domain of $2x^2 + 3x - 1$.

   solution: Here x is allowed to take any value, so the domain is all real numbers, $\mathbb{R}$.
5. Find the domain of $\dfrac{x}{(x-2)(x-3)}$

   solution: Here, x is allowed to take any value except 2 and 3 since both of these would make the bottom zero. Hence the domain is "all real numbers except 2 and 3" or, written in set notation, $\{x \mid x \neq 2, x \neq 3\}$.
6. Find the domain of $\dfrac{\sqrt{x}}{(x-5)}$

   solution: The square root makes it so we can't let x be any negative number, and the $x - 5$ in the bottom makes it so that x cannot be 5, so in set notation our domain is $\{x \mid x \neq 5 \text{ and } x \geq 0\}$.

Simplifying Rational Expressions

Just like when dealing with regular fractions, you can cancel a common term from the numerator and denominator as long as it is not zero:

$$\frac{A\cancel{C}}{B\cancel{C}} \text{ if } C \neq 0$$

Example:

$$\frac{x^2-1}{x^2+x-2} = \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x+2)} = \frac{x+1}{x+2} \qquad \text{if } x \neq 1$$

Multiplying and Dividing Rational Expressions

Multiplying and dividing rational expressions is done in an almost identical way to multiplying and dividing ordinary fractions.

Multiplication: $\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$

Example:

$$\frac{x^2+2x-3}{x^2+8x+16} \cdot \frac{x+4}{x-1} = \frac{(x-1)(x+3)}{(x+4)^2} \cdot \frac{x+4}{x-1} = \frac{x+3}{x+4}$$

Division: $\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$

Example:

$$\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} = \frac{x-4}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-3x-4} = \frac{\cancel{x-4}}{(x-2)\cancel{(x+2)}} \cdot \frac{(x+3)\cancel{(x+2)}}{\cancel{(x-4)}(x+1)} = \frac{x+3}{(x-2)(x+1)}$$

Adding and Subtracting Rational Expressions

To add or subtract rational expressions, they need to have the same denominator. If they do not, you must first put them under the same denominator before adding or subtracting them:

$$\frac{A}{B} + \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{D} + \frac{C}{D} \cdot \frac{B}{B} = \frac{AD+BC}{BD}$$

Examples:

1.

$$\begin{aligned} \frac{3}{x-1} + \frac{x}{x+2} &= \frac{3(x+2)}{(x-1)(x+2)} + \frac{x(x-1)}{(x+2)(x-1)} \\ &= \frac{3x+6+x^2-x}{(x-1)(x+2)} \\ &= \frac{x^2+2x+6}{(x+2)(x-1)} \end{aligned}$$

2.

$$\begin{aligned}\frac{1}{x^2-1}-x^2 &= \frac{1}{x^2-1}-\frac{x^2(x^2-1)}{x^2-1}\\ &= \frac{1-x^4+x^2}{x^2-1}\\ &= \frac{-x^4+x^2+1}{x^2-1}\end{aligned}$$

3. Simplify the expression $\dfrac{\frac{x}{y}+1}{1-\frac{y}{x}}$.

<u>*solution:*</u>

$$\frac{\frac{x}{y}+1}{1-\frac{y}{x}}=\frac{\frac{x+y}{y}}{\frac{x-y}{x}}=\frac{x+y}{y}\cdot\frac{x}{x-y}=\frac{x(x+y)}{y(x-y)}$$

4. Simplify the expression $\dfrac{\frac{1}{a+h}-\frac{1}{a}}{h}$.

<u>*solution:*</u>

$$\frac{\frac{1}{a+h}-\frac{1}{a}}{h}=\frac{\frac{a-(a+h)}{a(a+h)}}{h}=\frac{a-a-h}{a(a+h)}\cdot\frac{1}{h}=\frac{-1}{a(a+h)}$$

5. Simplify the expression $\dfrac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2}$

<u>*solution:*</u> We should first factor out $(1+x^2)^{-1/2}$ from the top, as it is the lowest power of that factor present.

$$\begin{aligned}\frac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2} &= \frac{(1+x^2)^{-1/2}((1+x^2)-x^2)}{1+x^2}\\ &= \frac{1+x^2-x^2}{(1+x^2)^{1/2}(1+x^2)}\\ &= \frac{1}{(1+x^2)^{1+1/2}}\\ &= \frac{1}{(1+x^2)^{3/2}}\end{aligned}$$

<u>Rationalizing the Denominator</u>

In order to rationalize a function with the denominator in the form $A+B\sqrt{C}$ we multiply the numerator and denominator by its **conjugate** $A-B\sqrt{C}$.

<u>Example:</u> $\dfrac{1}{1+\sqrt{2}}=\dfrac{1}{1+\sqrt{2}}\cdot\dfrac{1-\sqrt{2}}{1-\sqrt{2}}=\dfrac{1-\sqrt{2}}{(1-\sqrt{2})(1+\sqrt{2})}=\dfrac{1-\sqrt{2}}{(1-2)}=\dfrac{1-\sqrt{2}}{-1}=\sqrt{2}-1$

As a final note, here are some errors that commonly befall students. Be careful!

1. $(a+b)^2 \neq a^2 + b^2$
2. $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
3. $\sqrt{a^2+b^2} \neq a+b$
4. $\dfrac{1}{a} + \dfrac{1}{b} \neq \dfrac{1}{a+b}$
5. $\frac{a+b}{a} \neq b$
6. $a^{-1} + b^{-1} \neq (a+b)^{-1}$

1.5 Equations and Inequalities

Definition 1.13 *An* **equation** *is a statement that two mathematical expressions are equal. Solving an equation is to find all values for all the unknowns that make the equation true.*

An **inequality** *is a statement that two mathematical expressions are $<$, $>$, $\leq$, or $\geq$ each other. Solving an inequality is to find all values for the unknowns that make the inequality true.*

Examples:

1. $x+1=0$. The unknown is x and has one solution, $x=-1$.
2. Solve the equation $4x+7=19$.

   solution:

$$\begin{aligned} 4x+7 &= 19 \\ 4x &= 19-7 \\ \frac{4x}{4} &= \frac{12}{4} \\ x &= 3 \end{aligned}$$

Definition 1.14 *A* **linear equation** *in one variable is an equation equivalent to one of the form $ax+b=0$ where a and b are real numbers and x is the variable.*

Example: $4x-5=0$ is linear while $x^2+2x=8$ is not.

If $a=0$ solving a linear equation is either impossible (if $b \neq 0$) or trivial ($0=0$). If $a \neq 0$, then the solution of $ax+b=0$ is $x=-\frac{b}{a}$.

1. Solve the equation $7x-4=3x+8$.

solution: We isolate the variable by moving all the terms involving x to one side and everything else to the other side.

$$\begin{aligned} 7x-4 &= 3x+8 \\ 7x-3x &= 8+4 \\ 4x &= 12 \\ x &= \frac{12}{4} \\ x &= 3 \end{aligned}$$

2. Solve the inequality $4x+17 \leq 19$

   solution: We can solve inequalities in much the same way that we solve equalities:

$$\begin{aligned} 4x+17 &\leq 19 \\ 4x &\leq 19-17 \\ 4x &\leq 2 \\ x &\leq \frac{1}{2} \end{aligned}$$

   So the set of x that satisfy the inequality is all $x \leq \frac{1}{2}$. In set notation, the solution is $\{x \mid x \leq \frac{1}{2}\}$ and in interval notation the solution is $(-\infty, \frac{1}{2}]$.

Note: we solve inequalities in much the same way as equations with one key difference - multiplying each side of an inequality by the same **negative** number reverses the direction of the inequality.

Example: Solve the inequality $3x < 9x+4$.

solution:

$$\begin{aligned} 3x &< 9x+4 \\ 3x-9x &< 4 \\ -6x &< 4 \\ \left(-\frac{1}{6}\right)(-6x) &> 4\left(-\frac{1}{6}\right) \\ x &> -\frac{2}{3} \end{aligned}$$

Again, the set of all solutions can be written as $\{x \mid x > -\frac{2}{3}\}$ or $(-\frac{2}{3}, \infty)$.

Rules of Inequalities

1. $A \leq B \Leftrightarrow A + C \leq B + C$.
2. $A \leq B \Leftrightarrow A - C \leq B - C$.
3. If C is positive and $A \leq B$, then $AC \leq BC$ and vice-versa.
4. If C is negative and $A \leq B$, then $AC \geq BC$ and vice-versa.
5. If A and B are both positive and $A \leq B$, then $\frac{1}{A} \geq \frac{1}{B}$ and vice-versa.
6. If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$.

Definition 1.15 *A* **quadratic equation** *is an equation of the form $ax^2+bx+c=0$ where a, b and c are real numbers with $a \neq 0$. Solutions of this type of equation are called the* **zeroes** *or* **roots** *of the polynomial $ax^2 + bx + c$.*

In cases where the quadratic can be factored, finding the solutions is relatively easy. For example, if we wanted to solve $x^2 + 5x + 6 = 0$. We would notice that $x^2 + 5x + 6 = (x + 3)(x + 2)$ and hence our equation is

$$(x + 3)(x + 2) = 0$$

Since we have two terms multiplied together to give 0, we know that one of the terms must be zero. Hence, either $x + 3 = 0$ or $x + 2 = 0$ or, in other words, $x = -2$ or -3. Thus the roots of $x^2 + 5x + 6$ are -2 and -3.

If the quadratic cannot easily be factored, then to find the solutions you should use the **quadratic equation**: if $ax^2 + bx + c = 0$, then the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In this case, we have that $ax^2 + bx + c$ must factor as follows:

$$ax^2 + bx + c = \left(x - \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\right)\left(x - \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\right)$$

Example: Solve $3x^2 - 5x - 1 = 0$.

solution: $a = 3$, $b = -5$ and $c = -1$, so we have

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} = \frac{5 \pm \sqrt{25 + 12}}{6} = \frac{5 \pm \sqrt{37}}{6}$$

note that in this case there are two solutions, one from the plus and one from the minus.

The quantity $b^2 - 4ac$ that appears under the square root is called the **discriminant** and is denoted Δ.

- If $\Delta = b^2 - 4ac > 0$ then the equation has two distinct real solutions.
- If $\Delta = b^2 - 4ac = 0$ then the equation has exactly one real solution.
- If $\Delta = b^2 - 4ac < 0$ then the equation has no real solutions.

Example: Solve $x^2 + 2x - 1 = 0$.

solution: The discriminant Δ is

$$\Delta = b^2 - 4ac = (2)^2 - 4(1)(-1) = 8 > 0$$

so we have two solutions. They are

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm \sqrt{8}}{2}.$$

Since $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$, we have

$$x = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

Hence we also have that

$$x^2 + 2x - 1 = (x - (-1 + \sqrt{2}))(x - (-1 - \sqrt{2})) = (x + 1 - \sqrt{2})(x + 1 + \sqrt{2})$$

Non-Linear Inequalities

Sometimes you are asked to solve an inequality that is not linear. For example, suppose you are asked to find the x's for which $x^2 + 5x + 6 > 0$. The techniques mentioned earlier will not work, so we need something else. For our example, we know that $x^2 + 5x + 6$ factors, so that our inequality is

$$(x + 2)(x + 3) > 0$$

So we want to find the values of x that make the product of $(x+2)$ and $(x+3)$ positive. We know that for the product of two terms to be postitive, they must either both be positive or both be negative. With that in mind, we set up a table detailing where each of the factors is positive and negative:

x	$(-\infty, -3)$	-3	$(-3, -2)$	-2	$(-2, \infty)$
$x + 2$	$-$	$-$	$-$	0	$+$
$x + 3$	$-$	0	$+$	$+$	$+$
$(x + 2)(x + 3)$	$+$	0	$-$	0	$+$

The above is what is known as a **sign table**. The $x + 2$ line, for example, says that on $(-\infty, -3)$, at -3 and on $(-3, -2)$ $(x + 2)$ is negative while on $(-2, \infty)$ it is postitive, and so on. Thus to get the signs of the product function at the bottom of the table we multiply the pluses and minuses down the column. If there are no zeroes and an even number of $-$ signs, the result is $+$. If there are an odd number, the result is $-$.

From our table, we can see that $(x+2)(x+3)$ is positive on $(-\infty,-3)$ and on $(-2,\infty)$. Thus the solution set for $(x+2)(x+3)>0$ is

$$(-\infty,-3)\cup(-2,\infty)$$

In general, to solve non-linear inequalities, follow these steps:

- **Move all the terms to one side.** Rewrite the inequality so that all the terms are on one side. If there are multiple quotients, bring them over a common denominator.
- **Factor.** Factor the non-zero side as much as you can.
- **Find where each factor is zero.** Then split up your table at those values.
- **Draw your sign table.** Use test values in the intervals to fill in the signs.
- **Solve.** Use the table to determine where the inequality holds.

Note that it is very important to **move all the terms to one side.** Sign tables don't work if you don't do this first.

Examples:

1. Find the values of x such that $\frac{1+x}{1-x}\geq 1$.

   solution: We begin by moving everything to one side

$$\begin{aligned}
\frac{1+x}{1-x} &\geq 1\\
\frac{1+x}{1-x}-1 &\geq 0\\
\frac{1+x}{1-x}-\frac{1-x}{1-x} &\geq 0\\
\frac{1+x-(1-x)}{1-x} &\geq 0\\
\frac{2x}{1-x} &\geq 0
\end{aligned}$$

   So we have to split up our line at the points $x=0$ and $x=1$.

x	$(-\infty,0)$	$(0,1)$	$(1,\infty)$
x	$-$	$+$	$+$
$1-x$	$+$	$+$	$-$
$\frac{2x}{1-x}$	$-$	$+$	$-$

So this is postitive on $(0,1)$. Since we are solving $\frac{2x}{1-x} \geq 0$, we need to include any endpoints that make $\frac{2x}{1-x} = 0$. The only endpoint that does that is $x = 0$. Thus our solution is

$$S = [0,1)$$

2. (a) Solve $\dfrac{3}{x} + \dfrac{5}{x+2} = 2$.

   (b) Solve $\dfrac{3}{x} + \dfrac{5}{x+2} \geq 2$.

   <u>*solution:*</u>

   (a) We start by getting the left over a common denominator

$$\begin{aligned}
\frac{3}{x} + \frac{5}{x+2} &= 2 \\
\frac{3(x+2)+5x}{x(x+2)} &= 2 \\
3(x+2)+5x &= 2x(x+2) \\
8x+6 &= 2x^2+4x \\
0 &= 2x^2-4x-6 \\
0 &= x^2-2x-3 \\
0 &= (x-3)(x+1)
\end{aligned}$$

   Thus $x = 3$ or $x = -1$.

   (b) Here we have to be a bit more careful. We have to get everything on one side before doing anything.

$$\begin{aligned}
\frac{3}{x} + \frac{5}{x+2} &\geq 2 \\
\frac{3}{x} + \frac{5}{x+2} - 2 &\geq 0 \\
\frac{3(x+2)+5x}{x(x+2)} - 2 &\geq 0 \\
\frac{3(x+2)+5x-2(x(x+2))}{x(x+2)} &\geq 0 \\
\frac{-2x^2+4x+6}{x(x+2)} &\geq 0 \\
\frac{-2(x^2-2x-3)}{x(x+2)} &\geq 0 \\
\frac{-2(x-3)(x+1))}{x(x+2)} &\geq 0
\end{aligned}$$

   Thus we have 4 factors to consider, and must split our table up at the points $x = -2, -1, 0$, and 3.

x	$(-\infty,-2)$	$(-2,-1)$	$(-1,0)$	$(0,3)$	$(3,\infty)$
x	$-$	$-$	$-$	$+$	$+$
$x+1$	$-$	$-$	$+$	$+$	$+$
$x+2$	$-$	$+$	$+$	$+$	$+$
$-2(x-3)$	$+$	$+$	$+$	$+$	$-$
$\frac{-2(x-3)(x+1))}{x(x+2)}$	$-$	$+$	$-$	$+$	$-$

So $\frac{-2(x-3)(x+1))}{x(x+2)}$ is positive on $(-2,-1)$ and $(0,3)$. It is also 0 at the points $x=3$ and $x=-1$. Hence our inequality holds on

$$S=(-2,-1]\cup(0,3]$$

Absolute Value Inequalities

Sometimes we are asked to solve an inequality involving an absolute value. Usually this just reduces to solving two regular inequalities.

Inequality	**Equivalent Form**	**Interval**
$\lvert x\rvert < c$	$-c<x<c$	$(-c,c)$
$\lvert x\rvert \leq c$	$-c\leq x\leq c$	$[-c,c]$
$\lvert x\rvert > c$	$x<-c$ or $c<x$	$(-\infty,-c)\cup(c,\infty)$
$\lvert x\rvert \geq c$	$x\leq -c$ or $c\leq x$	$(-\infty,-c]\cup[c,\infty)$

Examples: Solve the inequalities.

1. $|x-5|<2$

   solution: From the chart we see this is equivalent to $-2<x-5<2$. We can solve this by adding 5 to each of the terms:

   $$-2<x-5<2$$

   $$3<x<7$$

   Thus the set of x that satisfies $|x-5|<2$ is the interval $(3,7)$. In general, if you are given an inequality of the form $|x-d|<c$, then the solution set is an interval centered around c with radius d.

2. $|3x+2|\geq 4$

   solution: From the chart we see this is equivalent to

   $$3x+2\leq -4 \text{ or } 4\leq 3x+2$$

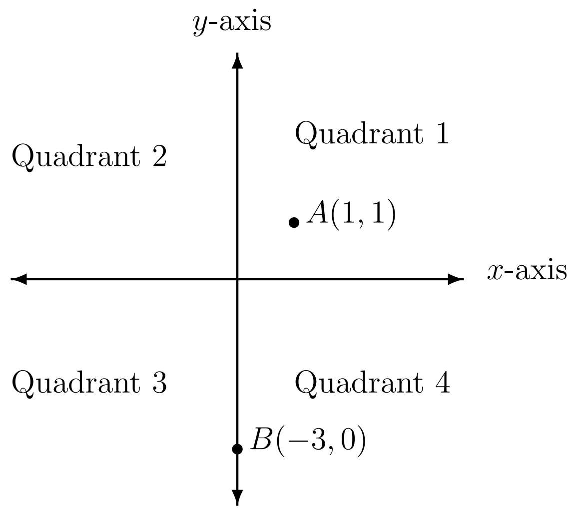
   $$3x\leq -6 \text{ or } 2\leq 3x$$

$$x \leq -2 \text{ or } \frac{2}{3} \leq x$$

$x \leq -2$ corresponds to the interval $(-\infty, -2]$ while $\frac{2}{3} \leq x$ corresponds to the interval $\left[\frac{2}{3}, \infty\right)$. Hence the solution set is $(-\infty, -2] \cup \left[\frac{2}{3}, \infty\right)$.

1.7 The Coordinate Plane

In this course we will be dealing a lot with the **Coordinate plane** (also called the *xy-plane*), so this section should serve as a review of it and its properties.



The xy-plane is divided into 4 quadrants by the x-axis and y-axis. Each point in the xy-plane is usually represented by a letter and an ordered pair $A(x, y)$. If we have two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the xy-plane, then:

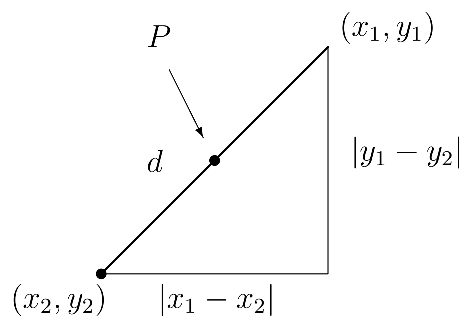
- the distance d between A and B is found using the Pythagorean theorem. The formula is:
$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
- the midpoint between A and B is found by averaging the x-coordinates and then averaging the y-coordinates. The formula is:
$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example: Let $A(1,2)$ and $B(3,0)$. The distance between A and B is

$$d(A,B) = \sqrt{(3-1)^2 + (2-0)^2} = \sqrt{2+2} = \sqrt{4} = 2$$

The midpoint between A and B is

$$\left(\frac{3+1}{2}, \frac{2+0}{2}\right) = (2,1)$$

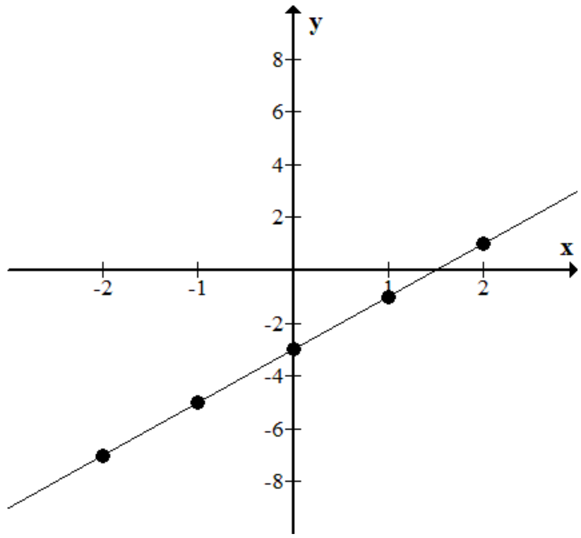


Graphs of Equations in Two Variables

Example: Sketch the graph of the equation $y = 2x - 3$.

solution: To see what this graph looks like, we can find some points on it

x	$y = 2x - 3$	(x, y)
-2	$y = 2(-2) - 3 = -7$	$(-2, -7)$
-1	$y = 2(-1) - 3 = -5$	$(-1, -5)$
0	-3	$(0, -3)$
1	-1	$(1, -1)$
2	1	$(2, 1)$



In general, the graph of an equation of the form $y = mx + b$ will be a straight line with slope m passing through the y-axis at $y = b$.

Intercepts

- x-intercepts - place or places where the graph crosses the x-axis. There could be several or none. To find the x-intercept, set $y = 0$ and solve for x.
- y-intercept - where graph crosses the y-axis. To find, set $x = 0$.

Example: Find the intercepts for $y = x^2 - 2$.

solution:

- We set $y = 0$ to find the x-intercept:

$$0 = x^2 - 2$$

$$x^2 = 2$$

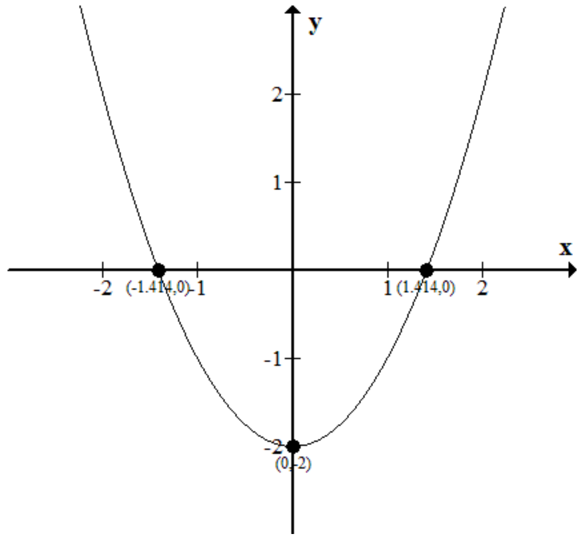
$$x = \pm\sqrt{2}$$

  So the graph of $y = x^2 - 2$ has two x-intercepts, $x = \sqrt{2}$ and $x = -\sqrt{2}$.

- We set $x = 0$ to find the y-intercept:

$$y = 0 - 2 = -2$$

  So the y-intercept is $y = -2$.



Equation of a Circle

Definition 1.16 *The equation of the circle with center* (h, k) *and radius* r *is*

$$(x-h)^2 + (y-k)^2 = r^2$$

Example: Sketch the graphs of $x^2 + y^2 = 25$ and $(x-2)^2 + (y+1)^2 = 25$.

solution: Both have radius 5 but the first is centered at $(0, 0)$ while the second is centered at $(2, 1)$.