Complex Numbers

Notation: $\mathbb{R}$ real numbers ; Preliminary story.

Def 1.1 Expressions of the form $a+bi$, where $a, b \in \mathbb{R}$ and $i^2=-1$ are called <u>complex numbers</u>. The set of all complex # is denoted $\mathbb{C}$.

Def 1.2 Given a complex # $z=a+ib$, the real # a is called <u>the real part</u> of z, and b is <u>the imaginary part</u> of z. Complex # of the form $i\cdot b = 0+ib$ are called <u>pure imaginary #</u>. The number $i=1i$ is called the imaginary <u>unit</u>. Observe that i <u>is not</u> a real #.

Multiplication & Addition

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1.3 Complex # are added and multiplied as follow:

If $z=a+ib$ and $w=a'+ib'$, then

$$z+w=(a+a')+i\cdot(b+b') \text{ and}$$

$$z\cdot w=(a+ib)\cdot(a'+ib')=aa'+i^2bb'+i\cdot ab'+i\cdot ba'=$$

$$=(aa'-bb')+i\cdot(ab'+ba')$$

Example 1.4 If $z=1+i$ and $w=2-3i$, compute $z^2+z\cdot w$

Solution $z^2=(1+i)^2=1+2i+i^2=1+2i-1=2i$

$$(a+b)^2=a^2+2ab+b^2$$

$$z\cdot w=(1+i)\cdot(2-3i)=(2+3)+i(2-3)=$$

$$=5-i$$

$$z^2+z\cdot w = 2i+5-i=5+i$$

Answer $z^2+z\cdot w=5+i$

Equality of complex #

$\boxed{\text{Lemma 1.5}}$ Two complex # are equal if and only if they have the same real and imaginary part. (3)

That is $a + ib = a' + i\cdot b' \iff a = a'$ and $b = b'$

Proof. $\Leftarrow$: Obvious

$\Rightarrow$ We prove by a contradiction

Let $a + ib = a' + ib'$. Assume that $\boxed{b \neq b'}$ then

$a - a' = i(b' - b)$

$\dfrac{a-a'}{b'-b} = i$ is REAL #, a contradiction.

Hence, $b = b'$

Assume that $\boxed{a \neq a'}$ then

$a - a' \neq 0$, a contradiction since $a - a' = i(b'-b) = i\,0 =$

Hence, $a = a'$

The proof is finished ∎

$\boxed{\text{Def 1.6}}$ If $z = a + ib$ is any complex #, the conjugate of z is another complex #, denoted $\bar{z}$, given by (4)

$$\bar{z} = a - ib$$

$\boxed{\text{Lemma 1.7}}$ The following properties hold:

C1 $\overline{(z \pm w)} = \bar{z} \pm \bar{w}$

C2 $\overline{(zw)} = \bar{z}\cdot\bar{w}$

C3 $\overline{(\bar{z})} = z$

C4 z is real if and only if $\bar{z} = z$

Proof. We prove only C4:

if z is real, then $z = a + i\cdot 0 = a$ and $\bar{z} = a - i\cdot 0 = a$

Hence $z = a = \bar{z}$

if $z = \bar{z}$, then $a + ib = a - ib$

$i\cdot(2b) = i(b+b) = a - a = 0$

By Lemma 1.5 $2b = 0$

Hence $b = 0$. i.e. $z = a + i\cdot 0$ is real ∎

Def. 1.8 Given any complex # $z = a + ib$, the **absolute value** or **modulus** of z is given by
$$|z| = \sqrt{a^2 + b^2}$$ (we take the positive square root only)

Lemma 1.9 For any complex # z we have:

A0 $\quad z\bar{z} = a^2 + b^2$

A1 $\quad |z| \geq 0$ for all z

A2 $\quad |z| = 0$ if and only if $z = 0$

A3 $\quad |zw| = |z||w|$

Proof of A3 $\quad |zw|^2 \underset{\text{by A0}}{=} (zw)\cdot\overline{(zw)} \underset{\substack{\text{by Lemma 1.7} \\ \text{prop C2.}}}{=} zw \cdot \bar{z}\bar{w} = z\cdot\bar{z} \cdot w\bar{w} \underset{\substack{\text{by} \\ \text{A0}}}{=} |z|^2 \cdot |w|^2$

Taking the square root we obtain the required equality ∎

(6)

Def. 1.10 If z is any non-zero complex #, the inverse of z denoted z^{-1}, is a complex # such that $z\cdot z^{-1} = z^{-1}\cdot z = 1$

Lemma 1.11 Given any non-zero complex # z, we have
$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

Proof. We have to check that $z\cdot z^{-1} = 1$
$$z z^{-1} = z \cdot \frac{\bar{z}}{|z|^2} = \frac{z\cdot\bar{z}}{z\cdot\bar{z}} = 1$$
by A0. of Lemma 1.9. ∎

Example 1.12 If $z = 2 + 3i$, to find z^{-1}

Solution: if $z = a + ib$, then $|z|^2 = a^2 + b^2$

Hence, for $z = 2 + 3i$ $\quad |z|^2 = 4 + 9 = 13$

So $\quad z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{1}{13}(2 - 3i) = \frac{2}{13} - \frac{3}{13}i$

(7)

1.13 Consider the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are real #-s with $a \neq 0$. It has roots given by the following formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } \sqrt{b^2-4ac} = d \text{ is called}$$

the discriminant.

1.14 The discriminant d determines the nature of the roots. Namely,

If $d > 0$ there are two real roots
If $d = 0$ there is one real root
If $d < 0$ there are no real roots

(8)

Def. 1.15 If the discriminant $d < 0$, then the quadratic $ax^2 + bx + c$ is said to be irreducible. ∎

Lemma 1.16 (a) Any irreducible quadratic has two nonreal complex roots which are conjugates of each other.
(b) Given any non-real complex # λ, then λ and $\bar{\lambda}$ are the complex roots of some real irreducible quadratic.

Proof. (a): the quadratic formula gives two complex roots

$$\lambda = \frac{1}{2a}\left(-b + i\sqrt{|d|}\right) \text{ and } \bar{\lambda} = \frac{1}{2a}\left(-b - i\sqrt{|d|}\right)$$

(b) The quadratic $x^2 - (\lambda + \bar{\lambda})x + (\lambda\bar{\lambda}) = (x-\lambda)(x-\bar{\lambda})$ has complex roots λ an[d $\bar{\lambda}$], is irreducible and has real coefficients $\lambda + \bar{\lambda}$ and $\lambda\bar{\lambda}$. ∎

Theorem 1.17 Every non constant polynomial with complex coefficients has a root.

Corollary 1.18 Every complex polynomial $f(x)$ of degree $n \geq 1$ has the form $f(x) = u \cdot (x-\lambda_1)(x-\lambda_2)\dots(x-\lambda$
where the complex #-s $\lambda_1, \lambda_2, \dots, \lambda_n$ are the roots of $f(x)$, and $u \neq 0$ is the coefficient of x^n in $f(x)$

Proof of 1.18 If $f(x)$ is a polynomial with complex coefficients of degree $n \geq 1$, and if λ_1 is a roo
then the factor theorem implies that
$f(x) = (x-\lambda_1)\,g(x)$, where $g(x)$ is a polynomial of degree $n-1$. Repeating this process for $g(x)$ and λ_2 a root of $g(x)$ and so on, we obtain the desired factorization. ☒