School of Mathematics and Statistics
Carleton University
Math. 1005A, Winter 2011

TEST 3

Any non-programmable calculator permitted, 1 blank sheet permitted for roughs

Print Name : Solutions

Student Number:

Tutorial Section (A1, A4, ...):

PART I: Multiple Choice Questions

(Choose and CIRCLE only ONE answer - No part marks here.)

1. **[2 marks]** Which of the following differential equations is of Cauchy-Euler type?
(a) $x^2y'' - y' + y = 0$, (b) $x^2y'' + 4xy' - y = 0$, (c) $xy'' + 4x^2y' + 6y = 0$, (d) $8x^2y'' + 3y' - x^2y = 0$.

2. **[2 marks]** The substitution $x = e^t$, $y(x) = z(t)$, transforms the differential equation of Cauchy-Euler type
$$4x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 4y = 0$$
into a differential with constant coefficients. Determine the auxiliary equation of this constant coefficient equation.
(a) $2\lambda^2 - 3\lambda + 5 = 0$, (b) $4\lambda^2 - 4\lambda + 1 = 0$, (c) $\lambda^2 + 5\lambda + 15 = 0$, (d) $2\lambda^2 - \lambda + 2 = 0$.

3. **[2 marks]** Find the general solution $y(x)$ of the differential equation
$$4x^2y'' + y = 0,$$
for $x > 0$.
(a) $y(x) = c_1\sqrt{x} + c_2\sqrt{x}\ln x$, (b) $y(x) = c_1x + c_2\sqrt{x}\ln x$, (c) $y(x) = c_1\sqrt{x} + c_2\ln x$, (d) $y(x) = c_1x^2 + c_2\sqrt{x}\ln x$.

4. **[2 marks]** Find a particular solution of the system of differential equations
$$x' = -6x + 2y, \qquad y' = -3x + y,$$
satisfying the initial condition $x(0) = 1, y(0) = 3$, where the solutions are functions of t.
(a) $x(t) = e^{-5t}$, $y(t) = 3$, (b) $x(t) = 1$, $y(t) = 3$
(c) $x(t) = 2$, $y(t) = 3e^{-5t}$, (d) $x(t) = 1$, $y(t) = 3 + te^{-5t}$.

5. **[2 marks]** Answer TRUE or FALSE:
The sequence $a_n = (-5)^n$, $n = 0, 1, 2, 3, \ldots$ converges to 5 as $n \to \infty$.
(a) TRUE, (b) FALSE

PART II: Show all work here and give details. No additional pages will be accepted

6. **[10 marks]** a) Evaluate $\lim_{n\to\infty} 3\frac{\sqrt{2n+1}}{\sqrt{n-1}}$ using any method.

(2) (2) (2)

$$\frac{3\sqrt{2n+1}}{\sqrt{n-1}} = 3\sqrt{\frac{2n+1}{n-1}} = 3\sqrt{\frac{n(2+\frac{1}{n})}{n(1-\frac{1}{n})}} = 3\sqrt{\frac{2+1/n}{1-1/n}}.$$

So as $n \to \infty$ $\quad \frac{3\sqrt{2n+1}}{\sqrt{n-1}} \to 3\sqrt{2} \leftarrow (4)$

or can use L'Hospital's Rule.

7. [**10 marks**] Find the general solution of the following system of differential equations using any method whatsoever:

$$x' = 10x - 5y, \qquad y' = 8x - 12y.$$

The matrix $A = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix} \leftarrow 1$

Eigenvalues: $\det(A - \lambda I) = 0$. ⓐ $\begin{vmatrix} 10-\lambda & -5 \\ 8 & -12-\lambda \end{vmatrix} = 0$ ← 1

$\Leftrightarrow (\lambda - 10)(\lambda + 12) + 40 = 0 \Rightarrow \boxed{\lambda^2 + 2\lambda - 80 = 0} \leftarrow 1$

$\therefore \boxed{\lambda = 8, \ \lambda = -10}$

1 1

$\lambda = 8$: $\therefore$ Eigenvectors: $\begin{bmatrix} 10-8 & -5 \\ 8 & -12-8 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or.

$\begin{bmatrix} 2 & -5 \\ 8 & -20 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \therefore \boxed{\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}}$ ← 1 or any multiple is OK, too.

$\lambda = -10$: Eigenvector. $\begin{bmatrix} 20 & -5 \\ 8 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}}$ ← 1

$$\boxed{\therefore \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-10t}}$$

1/2 1/2 1/2 1/2 1/2 1/2

Give 7/10 if eigenvalues incorrect but procedure is correct.