School of Mathematics and Statistics
Carleton University
Math. 1005A, Winter 2011

TEST 1

Any non-programmable calculator permitted, 1 blank sheet permitted for roughs

Print Name : SOLUTIONS

Student Number:

Tutorial Section (A1, A4, ...):

PART I: Multiple Choice Questions

(Choose and CIRCLE only ONE answer - No part marks here.)

1. c
2. a
3. b
4. b
5. a

1. **[2 marks]** Find the solution of the differential equation $\frac{dy}{dx} = \frac{x}{2y}$ subject to the initial condition $y(0) = 1$.

(a) $y = \frac{x^3}{3} + C$, (b) $y^2 = \frac{x^2}{3} - 1$, (c) $y^2 = \frac{x^2}{2} + 1$, (d) $y^2 = x^2 + 1$.

2. **[2 marks]** Find an integrating factor of the linear differential equation

$$\frac{dy}{dx} + \frac{x}{1+x^2} y = 1.$$

(a) $\sqrt{1+x^2}$, (b) $1 + x^2$, (c) e^{1+x^2}, (d) $(1+x^2)^{1/3}$

3. **[2 marks]** Find the general solution $y(x)$ of the differential equation

$$x^2 y' + x y = 1,$$

for $x > 0$.

(a) $y = \ln x + 2$, (b) $xy = \ln x + C$, (c) $y = x \ln x - C$, (d) $x^2 y = \ln x - 4$.

4. **[2 marks]** Find the solution $y(x)$ of the initial value problem for the differential equation $y' + (x-1)\, y = 0 \quad y(0) = 1$. Calculate the value of this solution at the point $x = 1$, i.e., find $y(1)$.

(a) $1 - e$, (b) $\sqrt{e}$ (c) e, (d) 0

5. **[2 marks]** Answer TRUE or FALSE: The substitution $v = y/x$ (for $x \neq 0$) transforms the differential equation $(y^2 + xy)\, dx - x^2\, dy = 0$ into the separable equation $v^3\, dx - x\, dv = 0$.

(a) TRUE, (b) FALSE

PART II: Show all work here and give details. No additional pages will be accepted

6. **[10 marks]** a) Find the solution of the differential equation $y' + (\sin x)\, y = e^{\cos x}$ satisfiying the initial condition $y(\pi/2) = 0$.

(is a First order linear de.)

Integrating factor given by $e^{\int \sin x\, dx} = e^{-\cos x}$ ← (2)

$\therefore \; y' e^{-\cos x} + \sin x\, e^{-\cos x} y = 1.$ ← (2)

or $\frac{d}{dx}\left(y e^{-\cos x}\right) = 1$ $\leftarrow$ (1)

$\boxed{y e^{-\cos x} = x + C}$ is general solution (2)

$\therefore y = (x+c)e^{\cos x}$.

Since $y(\frac{\pi}{2}) = 0 \Rightarrow \boxed{C = -\pi/2}$. $\leftarrow$ (2)

$\therefore \boxed{y = (x - \frac{\pi}{2})e^{\cos x}}$ $\leftarrow$ (1)

7. **[5+5 marks]** Solve the following differential equations for the required solution:

a) $(x+y)^2\,dx + (2xy + x^2 - 1)\,dy = 0 \qquad y(1) = 1$.

b) $xy' + y = x\,y^3$, (general solution for $x > 0$.)

(a) $M = (x+y)^2$, $N = 2xy + x^2 - 1 \Rightarrow \left.\begin{array}{l} M_y = 2(x+y) \\ N_x = 2y + 2x \end{array}\right\} \therefore$ $\leftarrow$ (1) $\leftarrow$ (1)

$M_y = N_x$ $\leftarrow$ (1)

$\therefore$ exact. & $\exists f \ni \frac{\partial f}{\partial x} = M$, $\frac{\partial f}{\partial y} = N$, i.e.,

$\therefore f(x,y) = \int M(x,y)\,dx + g(y) = \int (x+y)^2\,dx + g(y) = \boxed{\frac{(x+y)^3}{3} + g(y).}$ $\leftarrow$ (1)

Also $N = \frac{\partial f}{\partial y} = (x+y)^2 + g'(y) = 2xy + \cancel{x^2} - 1 \Rightarrow y^2 + g'(y) = -1$.

$g'(y) = -1 - y^2 \Rightarrow \boxed{g(y) = -y - \frac{y^3}{3}}$ $(\frac{1}{2})$

$\therefore$ general solution is $\frac{(x+y)^3}{3} - y - \frac{y^3}{3} = C$. & if $y(1) = 1$

$\Rightarrow \boxed{C = 4/3}$ $(\frac{1}{2})$

(b). Bernoulli equation with $n = 3$. $\leftarrow$ (1)

So let $z = y^{1-n} = y^{1-3} = \boxed{y^{-2} = z}$ $\leftarrow$ (1)

$\Rightarrow \boxed{z' - \frac{2}{x}z = -2}$ $\leftarrow$ (1)

i. factor $= 1/x^2$ $\leftarrow$ (1).

$\therefore$ general solution for $x > 0$ is given by

(1) $\rightarrow$ $\boxed{y^2(x) = \frac{1}{2x + Cx^2},}$ $C \in \mathbb{R}$, a constant.