1. **(2 marks each; total 8 marks)**

(a) What are the domains of the following 2 functions: (i) $f(x) = \sqrt{x+2}$ and (ii) $g(x) = \dfrac{1}{x^2 - x}$?

i) $x+2 \geq 0 \Rightarrow x \geq -2$ OR $[-2, \infty)$

ii) $x^2 - x \neq 0 \Rightarrow x(x-1) \neq 0 \Rightarrow x \neq 0, x \neq 1$

OR $(-\infty, 0) \cup (0,1) \cup (1, \infty)$

(b) Determine whether the function $f(x) = \dfrac{x^4 - 4x^2}{3}$ is even, odd, or neither.

$$f(-x) = \frac{(-x)^4 - 4(-x)^2}{3} = \frac{x^4 - 4x^2}{3} = f(x)$$

$\Rightarrow$ EVEN

(c) If $f(x) = 1 - x^3$ and $g(x) = \dfrac{1}{x}$ find $f(g(x))$ and $g(f(x))$.

$$f(g(x)) = 1 - \left(\frac{1}{x}\right)^3 = 1 - \frac{1}{x^3}$$

$$g(f(x)) = \frac{1}{1-x^3}$$

(d) Consider the function $f(x) = e^{x^4}$. Find the inverse $f^{-1}(x)$.

$y = e^{x^4}$

invert x+ y

$x = e^{y^4}$

$\Rightarrow f^{-1}(x) = [\ln(x)]^{1/4}$

solve for y

$\ln(x) = y^4$

$y = [\ln(x)]^{1/4}$

2. **(3 marks)** What do we mean when we say that $f(x)$ is continuous at $x = a$? There are three properties. List all three.

   (a) $f(a)$ is defined

   (b) $\lim_{x \to a} f(x)$ exists

   (c) $f(a) = \lim_{x \to a} f(x)$

3. **(Total 6 marks)** If a ball is dropped from the upper deck of the CN Tower with the given equation of motion $f(t) = 4.9t^2$ (where f is measured in meters and time t is measured in seconds), find:

   (a) **(2 marks)** The average velocity of the ball between 2 and 3 seconds.

   $f(2) = 4.9(2)^2 = (4.9)(4)$, $f(3) = 4.9(3)^2 = (4.9)(9)$

   $$\text{avg}\,v = \frac{f(3) - f(2)}{3-2} = \frac{4.9(9) - 4.9(4)}{1} = \frac{(4.9)5}{1} = 24.5 \text{ m/s}$$

   (b) **(4 marks)** The instantaneous velocity at $t = 5$ seconds. [Hint: instantaneous rate of change is found by $\lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$]

$$V = \lim_{t \to 5} \frac{f(t) - f(5)}{t-5} = \lim_{t \to 5} \frac{4.9t^2 - 4.9(5)^2}{t-5} = \lim_{t \to 5} \frac{4.9(t^2 - 5^2)}{t-5}$$

$$= \lim_{t \to 5} \frac{4.9\cancel{(t-5)}(t+5)}{\cancel{(t-5)}}$$

$$= \lim_{t \to 5} 4.9(t+5)$$

$$= 4.9(5+5)$$

$$= 49 \text{ m/s}$$

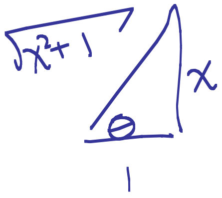
4. **(4 marks)** Find $\sec\left(\tan^{-1} 3x\right)$.

$\tan^{-1}(x) = \theta$

$\Rightarrow \tan(\theta) = \frac{x}{1} = \frac{o}{a}$

by Pythag..

↳



so $\sec(\tan^{-1}(3x))$

$= \sec(\theta)$

$= \frac{1}{\cos(\theta)}$

$= \frac{h}{a}$

$= \frac{\sqrt{x^2+1}}{1}$

$= \sqrt{x^2+1}$

5. **(5 marks)** Sketch a function satisfying ALL of the following properties:

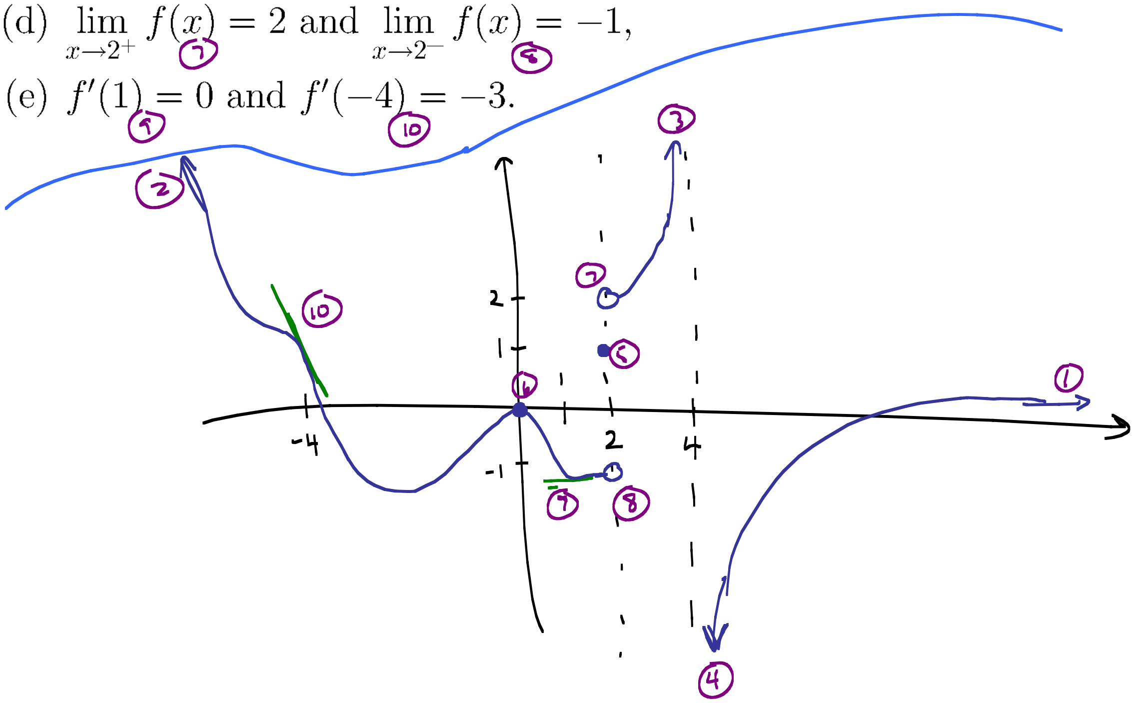
(a) $\lim_{x\to+\infty} f(x) = 0$ ① and $\lim_{x\to-\infty} f(x) = +\infty$ ②,

(b) $\lim_{x\to 4^-} f(x) = +\infty$ ③ and $\lim_{x\to 4^+} f(x) = -\infty$ ④,

(c) $f(2) = 1$ ⑤ and $f(0) = 0$ ⑥,

(d) $\lim_{x\to 2^+} f(x) = 2$ ⑦ and $\lim_{x\to 2^-} f(x) = -1$ ⑧,

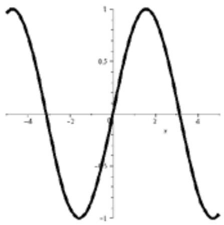
(e) $f'(1) = 0$ ⑨ and $f'(-4) = -3$ ⑩.



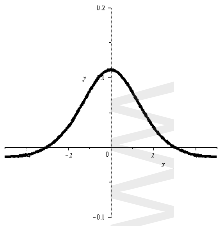
6. **(6 marks)** Match the following graphs of the function $f(x)$ with their corresponding graph of the derivative $f'(x)$

$f(x)$ $f'(x)$

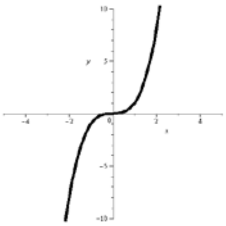
(a)



F



(b)



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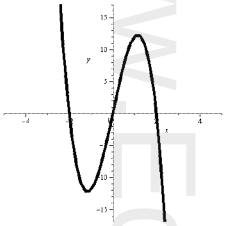
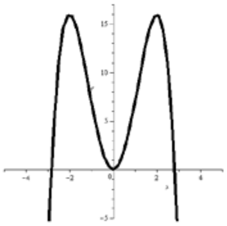


(c)



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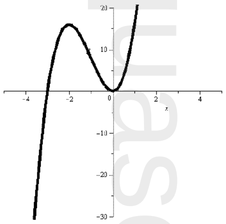
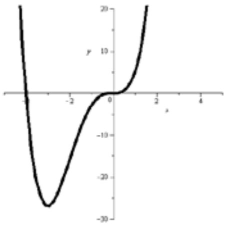


(d)



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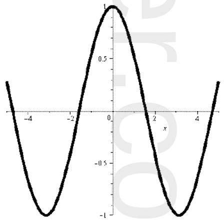
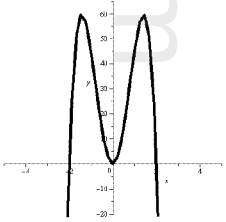
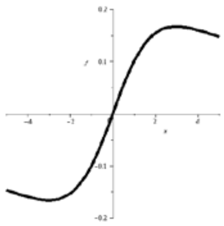


(e)

E



(f)



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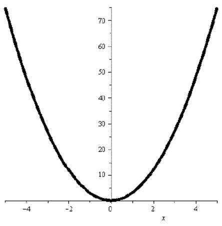

7. **(2 mark each; total 8 marks)** Evaluate the following limits.

(a) $\lim\limits_{x\to4}\dfrac{x^2-4x}{x^2-3x-4}$ $= \lim\limits_{x\to4}\dfrac{x\cancel{(x-4)}}{\cancel{(x-4)}(x+1)} = \dfrac{4}{5}$

(b) $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-1}{x}$ $\cdot\dfrac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$$= \lim_{x\to0}\frac{(\sqrt{1+x})^2-(1)^2}{x(\sqrt{1+x}+1)}$$

$$= \lim_{x\to0}\frac{\cancel{1}+x-\cancel{1}}{x(\sqrt{1+x}+1)}$$

$$= \lim_{x\to0}\frac{\cancel{x}}{\cancel{x}(\sqrt{1+x}+1)}$$

$$= \frac{1}{\sqrt{1+0}+1}$$

$$= \frac{1}{2}$$

(c) $\lim\limits_{x\to\infty}\dfrac{5+2x-3x^2}{x^2+10x-3}$ $\cdot\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$$= \lim_{x\to\infty}\frac{\overset{\to 0}{\frac{5}{x^2}}+\overset{\to 0}{\frac{2}{x}}-3}{1+\underset{\to 0}{\frac{10}{x}}-\underset{\to 0}{\frac{3}{x^2}}}$$

$$= \frac{-3}{1} = -3$$

(d) $\lim\limits_{x\to\infty}(x^2-x)$

$$= \lim_{x\to\infty} x(x-1)$$

$$= \infty\cdot\infty$$

$$= \infty$$

8. **(6 marks)** Given the following function

$$f(x)=\begin{cases} ax^2+1, & x<1\\ 5-bx, & 1\le x<3\\ -2, & x\ge 3.\end{cases}$$

Determine the values of a and b that make the function continuous.

NEED $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$: $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} -2 = -2$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} 5-bx = 5-b(3) = 5-3b$

so we need $-2 = 5-3b$

$\boxed{b = 7/3}$

$\boxed{\text{since } b=\frac{2}{3}}$

NEED $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 5-bx = 5-\left(\frac{2}{3}\right)(1) = 5-\frac{2}{3} = \frac{8}{3}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} ax^2+1 = a(1)^2+1 = a+1$

so we need $\frac{8}{3} = a+1 \Rightarrow \boxed{a = 5/3}$

9. **(4 marks)** Use the Intermediate Value Theorem to find an interval which contains a root of the equation

$$\frac{1}{5}e^x = 2-x.$$

let $f(x) = \frac{1}{5}e^x - 2 + x$ (Note $f(x) = 0 \Leftrightarrow \frac{1}{5}e^x = 2-x$)

$f(x)$ has a domain of all real numbers and $f(x)$ is continuous on its domain, therefore IVT applies

when $x=0$, $f(0) = \frac{1}{5}e^0 - 2 + (0) = \frac{1}{5} - 2 < 0$

when $x=2$, $f(2) = \frac{1}{5}e^2 - 2 + 2 = \frac{1}{5}e^2 > 0$

since $f(0) < 0$ and $f(2) > 0$, by IVT the is a number $c \in (0,2)$ such that $f(c) = 0$

(space for rough work)