MATH2004A – Test 1 – 1:35 pm - 2:25 pm, Oct. 3
Name: **Student Number:**
Total: 16 marks (for 4 questions). You may write on **both sides**.

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1. [6 points] Let

$$f(x) = \begin{cases} 1, & \text{for } x \in [0, \pi), \\ 0, & \text{for } x \in [\pi, 2\pi]. \end{cases}$$

(i) Let f_{odd} be the 4π-periodic odd extension of f, determine f_{odd} on the interval $(-2\pi, 0)$.
(ii) Find the Fourier sine series.
(iii) Assume the Fourier sine series converges to A at $x_1 = \pi$, and converges to B at $x_2 = (47\pi)/4$. Find A and B.

Solution: (i) [1 point]

$$f_{odd}(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 0 & \text{if } -2\pi < x \le -\pi. \end{cases}$$

(ii) [4 points subtotal] It is adequate to determine b_n. We have $2L = 4\pi$. So $L = 2\pi$ [0.5 point]. For $n \ge 1$,

$$b_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\sin\frac{n\pi x}{L}dx \quad [2\ points] \tag{1}$$

$$= \frac{2}{L}\int_{0}^{L} f_{odd}(x)\sin\frac{n\pi x}{L}dx \tag{2}$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin(nx/2)dx \tag{3}$$

$$= \frac{2}{n\pi}[1 - \cos(n\pi/2)]. \tag{4}$$

[1 point for final evaluation] If the final answer is correct, it is allowed to skip some of the steps.

The Fourier sine series is [0.5 point] $f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}[1 - \cos(n\pi/2)]\sin(nx/2)$.
(iii) [1 point]

$$\text{Fourier Series}(x_1) = \text{Fourier Series}(\pi) = (0 + 1)/2.$$

So $A = 0.5$ [0.5 point].

$$\text{Fourier Series}(x_2) = \text{Fourier Series}(47\pi/4 - 12\pi) = \text{Fourier Series}(-\pi/4) = -1.$$

So $B = -1$ [0.5 point].

2. [4 points] Consider the parametric curve C given by $x = 2t^2, \quad y = t^2 + 3t + 1, \quad 0 \le t \le 3$. Find the equation of the tangent line to the curve at $t = 1$.

Solution: We have $(x, y)|_{t=1} = (2, 5)$.
$dx/dt = 4t$ and $dy/dt = 2t + 3$. [1 point]
So the slope at $t = 1$ is [1.5 points]

$$dy/dx = (2t + 3)/4t = 5/4.$$

The tangent is $y - 5 = (5/4)(x - 2)$. [1.5 points]

3. [5 points] Consider the parametric curve C given by $x = e^t + 2e^{-t}, \quad y = (2\sqrt{2})t + 3, \qquad 0 \leq t \leq 1$. Find the length of C.

Solution. $dx/dt = e^t - 2e^{-t}$ [1 point]; and $dy/dt = 2\sqrt{2}$ [0.5 point].

$$L = \int_0^1 \sqrt{(dx/dt)^2 + (dy/dt)^2} dt \quad [2\ points] \tag{5}$$
$$= \int_0^1 \sqrt{e^{2t} + 4e^{-2t} - 4 + 8} dt \tag{6}$$
$$= \int_0^1 (e^t + 2e^{-t}) dt \tag{7}$$
$$= 1 + e - (2/e). \tag{8}$$

[1.5 points for evaluation]

4. [1 bonus point] (i) Plot the point P with polar coordinates $(\pi/4, -2)$. (ii) For the point Q with Cartesian coordinates $(-1/2, \sqrt{3}/2)$, find its polar coordinates.

Solution.
0.5 point for each part.
(ii) $(\theta, r) = (2\pi/3, 1)$.