Solution to Midterm 1 (A)

MAT2322, Winter 2012
Total = 20 marks

1. (3 marks) Find the angle between vectors $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (0, 1, 1)$.

Solution. $\theta = \arccos\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \arccos\dfrac{1}{2} = \dfrac{\pi}{3}$.

2. (4 marks) Consider two lines L_1: $(x, y\ z) = (1, 2, 3) + (2, 3, 2)\,t$, and L_2: $(x, y\ z) = (-1, -2, 4) + (4, 3, 1)\,s$.

(a) (1 mark) Show that these two lines are skew lines.

(b) (3 marks) Find the distance between these two lines.

Solution. (a) Equate x, y and z in these two lines: $1 + 2t = -1 + 4s$, $2 + 3t = -2 + 3s$, $3 + 2t = 4 + s$. From the last equation, $s = 2t - 1$. Substitute it into the first two equations: $4(2t - 1) - 2t = 2$, and $3(2t - 1) - 3t = 4$. The former gives $t = 1$, and the later gives $t = 7 / 3$. They are not consistent. Hence, these two lines do not have a common point.

(b) In parametric forms, the equations of these two lines are

$\mathbf{p}_1 = (1, 2, 3)$, $\mathbf{p}_2 = (-1, -2, 4)$, $\mathbf{v}_1 = (2, 3, 2)$, $\mathbf{v}_2 = (4, 3, 1)$.

The cross product $\mathbf{v}_1 \times \mathbf{v}_2$ is $\mathbf{n} = (-3, 6, -6)$.

The distance between L_1 and L_2 is the length of the projection of vector $\mathbf{r} = \mathbf{p}_1 - \mathbf{p}_2 = (2, 4, -1)$ onto $\mathbf{n}$:

$$d = \left|\frac{\mathbf{r}\cdot\mathbf{n}}{|\mathbf{n}|}\right| = \frac{24}{9} = \frac{8}{3}.$$

3. (4 marks) Consider the curve $\mathbf{r}(t) = (\sqrt{2}\,t, e^{-t}, e^{t})$, $-\infty < t < \infty$.

(a) (2 marks) Find the length of the segment of the curve with $0 \le t \le 1$.
(b) (2 marks) Find an equation of the tangent line of this curve at the point where $t = 0$.

Solution. (a) $\mathbf{r}'(t) = (\sqrt{2}, -e^{-t}, e^{t})$.

$$L = \int_0^1 \sqrt{2 + e^{-2t} + e^{2t}}\,dt = \int_0^1 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_{t=0}^1 = e - e^{-1}.$$

(b) The direction vector of the tangent line is $\mathbf{r}'(0) = (\sqrt{2}, -1, 1)$.

$\mathbf{r}(0) = (0, 1, 1)$. An equation of the tangent line is

$(x, y, z) = (0, 1, 1) + (\sqrt{2}, -1, 1)t$.

4. (5 marks) Consider the curve $\mathbf{r}(t) = \left(2t, t^2, \frac{1}{3}t^3\right)$.

(a) (1 mark) Find $\mathbf{r}'(t)$ and the length of $\mathbf{r}'(t)$.
(b) (2 marks) Find the curvature of the curve at the point where $t = 1$.
(c) (2 mark) Find an equation of the osculating plane of the curve at $t = 1$.

Solution. (a) $\mathbf{r}'(t) = (2, 2t, t^2)$, $|\mathbf{r}'(t)| = \sqrt{t^4 + 4t^2 + 4} = t^2 + 2$.

(b) *Method* 1. $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{t^2+2}(2, 2t, t^2)$.

$\mathbf{T}'(t) = \frac{1}{(t^2+2)^2}\left(-4t, 2(2-t^2), 4t\right)$. When $t = 1$, $\mathbf{T}'(1) = \frac{1}{9}(-4, 2, 4)$. $|\mathbf{T}'(1)| = \frac{2}{3}$.

Since $|\mathbf{r}'(1)| = 3$. $\kappa(1) = |\mathbf{T}'(2)| / |\mathbf{r}'(2)| = \frac{2}{9}$.

Method 2. $\mathbf{r}''(t) = (0, 2, 2t)$. $\mathbf{r}'(t) \times \mathbf{r}''(t) = (2t^2, -4t, 4)$. $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{4t^4 + 16t^2 + 16} = 2(t^2 + 2)$. The curvature is

$$\kappa(t) = \left[\frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}\right]_{t=1} = \frac{6}{3^3} = \frac{2}{9}.$$

(c) At the point where $t = 1$, $\mathbf{T}$ is in the direction of vector $\mathbf{t} = (2, 2, 1)$, and $\mathbf{T}'$ is in the direction of $\mathbf{n} = (-2, 1, 2)$. A normal vector of the osculating plane is $\mathbf{b} = \mathbf{t} \times \mathbf{n} = 3(1, -2, 2)$.
The osculating plane has the form $x - 2y + 2z = d$. Since $\mathbf{r}(1) = \left(2, 1, \frac{1}{3}\right)$, $d = 2 - 2 + \frac{2}{3} = \frac{2}{3}$.

The equation of the osculating plane is $x - 2y + 2z = \frac{2}{3}$.

5. (4 marks) Find all critical points of the function $z = 2x^3 + xy^2 + 5x^2 + y^2$, and use the second derivative test to classify each critical point as a local minimum, or a local maximum, or a saddle point.

Solution. $z_x = 6x^2 + y^2 + 10x$. $z_y = 2y(x + 1)$. $z_y = 0$ implies $y =$ or $x = -1$. If $y = 0$, from $z_x = 0$, $6x^2 + 10x = 0$, $x = 0$, or $x = -5 / 3$. If $x = -1$, from $z_x = 0$, we have $y^2 = 4$, $y = \pm 2$. There are four critical points: (0, 0), (–5/ 3, 0), (–1, –2), and (–1, 2).

$z_{xx} = 12x + 10$, $z_{yy} = 2x + 2$, $z_{xy} = 2y$. $D = z_{xx}z_{yy} - z_{xy}^2 = 2(6x + 5)(2x + 2) - 4y^2$.

$D(0, 0) > 0$, $z_{xx} > 0$, (0, 0) is a local minimum.
$D(0, -5 / 3) > 0$, $z_{xx} < 0$, (–5 / 3, 0) is a local maximum.
$D(-1, -2) = D(-1, 2) < 0$, (–1, –2) and (–1, 2) are saddle points.