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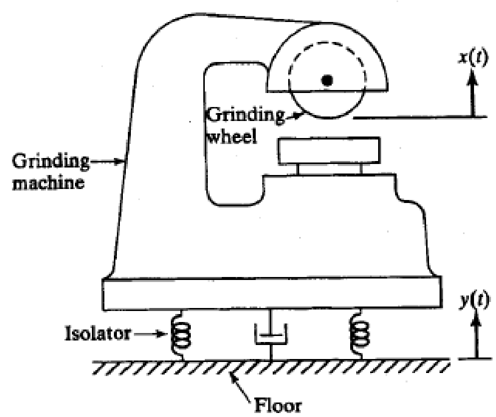
MCG 4308
Mechanical Vibration Analysis

MIDTERM EXAMINATION
February 17, 2012
SOLUTIONS

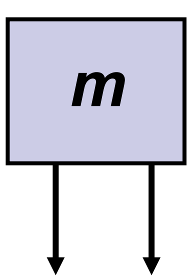
Question 1

A precision grinding machine is supported on an isolator consisting of two identical springs and a damper. The floor, on which the machine is mounted, is subjected to a harmonic disturbance due to the operation of an unbalanced pump that is located nearby.

a) Write the equation of motion for this system.
b) Since the motion of the base is harmonic, $\left(y(t) = Y\sin\left(\omega t\right)\right)$ calculate the **magnitude and phase** of the steady state response of the system.
c) What is the displacement transmissibility of the base motion?
d) It is discovered that the operating frequency of the unbalanced pump is close to the natural frequency of the grinding machine so that the amplitude of the motion of the grinding machine is too high. Which of the following techniques can be used, with confidence, to reduce the motion of the machine? Choose ALL that apply and explain.
   A. Reduce the mass of the machine
   B. Increase the mass of the machine
   C. Reduce the stiffness of the isolator (springs)
   D. Increase the stiffness of the isolator (springs)
   E. Reduce the damping of the isolator
   F. Increase the damping of the isolator



Solution



Equation: $m\ddot{x} + c\dot{x} + 2kx = c\dot{y} + 2ky$

$$\Rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{y} + \omega_n^2 y \qquad \omega_n^2 = \frac{2k}{m}$$

Sub into equation: $y(t) = Ye^{i\omega t} \qquad x(t) = Xe^{i(\omega t+\phi)} = Xe^{i\phi}e^{i\omega t}$

$$\left[-\omega^2 + 2\zeta\omega_n\omega\, i + \omega_n^2\right]Xe^{i\phi}e^{i\omega t} = \left[2\zeta\omega_n\omega\, i + \omega_n^2\right]Ye^{i\omega t}$$

$2k(x - y) \quad c(\dot{x} - \dot{y})$

Divide both sides by ω_n^2 and $e^{i\omega t}$,remember $r = \omega/\omega_n$

$$\left[-r^2 + 2\zeta r i + 1\right] X e^{i\phi} = \left[2\zeta r i + 1\right] Y$$

$$X e^{i\phi} = \frac{2\zeta r i + 1}{1 - r^2 + 2\zeta r i} Y$$

So match up magnitudes and phases:

$$X = \text{magnitude of} \left[\frac{2\zeta r i + 1}{1 - r^2 + 2\zeta r i}\right] Y = Y\sqrt{\frac{(2\zeta r)^2 + 1}{(1 - r^2)^2 + (2\zeta r)^2}}$$

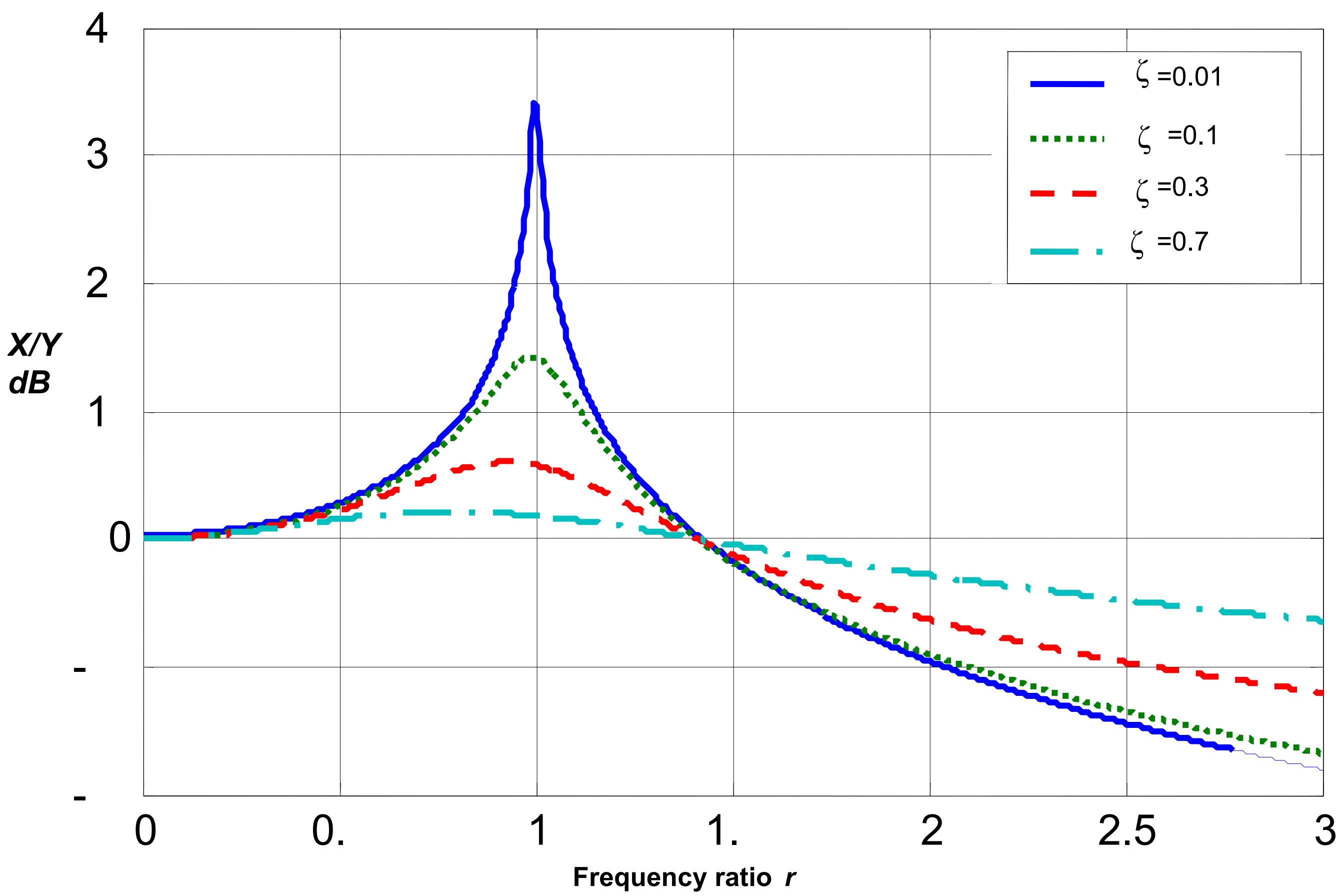
$$\phi = \text{phase of} \left[\frac{2\zeta r i + 1}{1 - r^2 + 2\zeta r i}\right] = \arctan\left(\frac{2\zeta r}{1}\right) - \arctan\left(\frac{2\zeta r}{1 - r^2}\right)$$

so the system response is $x(t) = X\sin(\omega t + \phi)$

c) The transmissibility ratio is give by

$$\frac{X}{Y} = \sqrt{\frac{(2\zeta r)^2 + 1}{(1 - r^2)^2 + (2\zeta r)^2}}$$

which is just the magnitude part of part (b) rearranged as a ratio of input response magnitude to input magnitude.

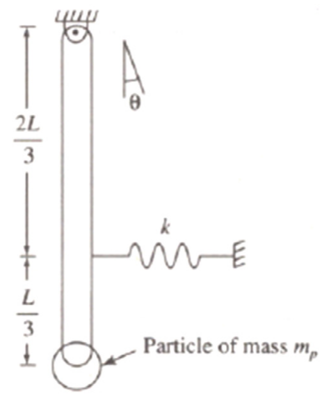


To decrease amplitude of response, need to increase damping or change the natural frequency of the system. Changing either the mass or stiffness will change the natural frequency of the system. So all situations except for (E) apply. We can change the natural frequency (change m or k) or increase the damping.

Question 2

Consider the system shown in the figure. Use θ, the counterclockwise angular displacement of the bar from the system's equilibrium position, as the coordinate and assume a small angular displacement. The moment of inertia of the bar about its centre of mass is given by $I_G = \frac{1}{12} mL^2$ where *m* is the mass of the bar. For this system find:

a) The equivalent mass
b) The equivalent stiffness
c) The natural frequency



The kinetic energy of the system at an arbitrary instant is

$$T = \tfrac{1}{2}m\bar{I}\dot{\theta} + \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}m_p v_p^{\,2}$$
$$= \tfrac{1}{2}(\tfrac{1}{12}mL^2)\dot{\theta}^2 + \tfrac{1}{2}m(\tfrac{1}{2}L\dot{\theta})^2 + \tfrac{1}{2}m_p(L\dot{\theta})^2$$
$$= \tfrac{1}{2}(\tfrac{1}{3}mL^2 + m_pL^2)\dot{\theta}^2$$

Using a horizontal plane through the pin support as the datum for potential energy calculations, the difference in potential energies between an arbitrary system position and the system's initial position is

$$V = \frac{1}{2}k\left(\frac{2}{3}L\theta\right)^2 + mg\frac{L}{2}(1 - \cos\theta) + m_p gL(1 - \cos\theta)$$

For small θ

$$\cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

Hence

$$V = \frac{1}{2}\left(\frac{4}{9}kL^2\right)\theta^2 + mg\frac{L}{2}\left(\frac{1}{2}\theta^2\right) + m_p gL\left(\frac{1}{2}\theta^2\right)$$
$$= \frac{1}{2}\left(\frac{4}{9}kL^2 + mg\frac{L}{2} + m_p gL\right)\theta^2$$

Since an angular coordinate was chosen as the generalized coordinate, the appropriate equivalent system is the torsional system with

$$I_{eq} = \frac{1}{3}mL^2 + m_pL^2 \qquad k_{t_{eq}} = \frac{4}{9}kL^2 + mg\frac{L}{2} + m_p gL$$

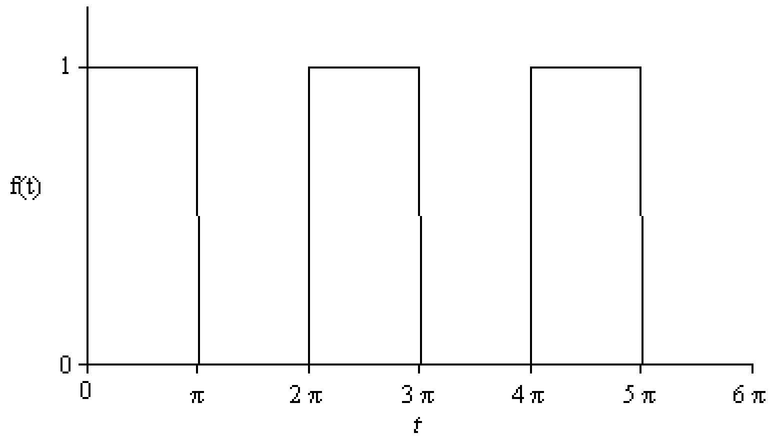
Natural frequency = sqrt(k_{eq}/ I_{eq})

Question 3

The motion of a spring-mass-damper system is described by the following equation

$$\ddot{x} + 2\dot{x} + 10x = f(t)$$

with $f(t)$ as shown in the figure below.



a) Find (the form of) the free vibration system response.
b) What are the natural frequency and damping ratio for this system? Is the system underdamped, critically damped or overdamped?
c) Find the Fourier series for the function shown in the figure above.
d) The forced response to $f(t)$ will be another Fourier series. For the purpose of computer simulations, how many terms in the Fourier series would you keep? Explain your answer.

Solution

Homogeneous Solution: $\ddot{x}_h + 2\dot{x}_h + 10x_h = 0$

Guess $x_h = e^{st}$ and sub into equation : $\left[s^2 + 2s + 10\right]e^{st} = 0$

characteristic equation: $s^2 + 2s + 10$, with roots: $s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 40}}{2} = -1 \pm 3i$

$\Rightarrow x_h(t) = A_1 e^{-t}\cos(3t) + A_2 e^{-t}\sin(3t)$

(Undamped) Natural frequency is given by $\sqrt{10} = 3.16$ rad/sec .
The damping ratio is given by c/2sqrt(k*m)

$$\frac{\sqrt{10}}{10}$$

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evalf(2/(2*sqrt(10)));
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$$0.3162277660$$

This system is clearly underdamped.

The square wave in the figure is described by

> $f(t) := \begin{cases} 1 & 0 \le t \le \pi \\ 0 & \pi < t \le 2\cdot\pi \end{cases}$

> $T := 2\cdot\pi : \quad \omega f := \frac{2\cdot\pi}{T} = 1$

$assume(n, integer):$

$$F(n) := \begin{cases} \frac{1}{T}\cdot\int_0^T f(t)\cdot\exp(-I\cdot n\cdot \omega f\cdot t)\,dt & n \ne 0 \\ \frac{1}{T}\cdot\int_0^T f(t)\,dt & n = 0 \end{cases}$$

$$F(n) = \begin{cases} -\frac{\frac{1}{2}\, I\,(1-(-1)^n)}{\pi\, n} & n \ne 0 \\ \frac{1}{2} & n = 0 \end{cases}$$

$simplify(F(n))$ assuming n, $even$

$$\begin{cases} \frac{1}{2} & n = 0 \\ 0 & otherwise \end{cases}$$

$simplify(F(n))$ assuming n, odd

$$-\frac{I}{\pi\, n}$$

$$F(n) = \begin{cases} \frac{1}{2} & n = 0 \\ \frac{-i}{\pi n} & n \text{ odd} \end{cases}$$

If we want to use the sin/cosine formulation

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nt) + b_n \sin(nt)\right]$$

$$a_0 = \frac{2}{2\pi}\int_0^{\pi} 1\, dt = \frac{2}{2\pi}\pi = 1$$

$$a_n = \frac{2}{2\pi}\int_0^{\pi} \cos\left(nt\right) dt = \frac{1}{\pi}\left.\frac{\sin(nt)}{\pi}\right|_0^{\pi} = 0$$

$$b_n = \frac{2}{2\pi}\int_0^{\pi} \sin\left(nt\right) dt = \frac{1}{\pi}\left.\frac{\cos\left(nt\right)}{-n}\right|_0^{\pi} = \frac{-1}{n\pi}\left(\cos(n\pi)-1\right) = \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{n\pi} & n \text{ odd} \end{cases}$$

$$f(t) = \frac{1}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{2}{n\pi}\sin(nt)$$

Need to keep enough terms so that AT LEAST

$$N\omega_f = N(1) \geq \omega_n = 3.16$$

So need to keep AT LEAST 4 terms (i.e. n=0,±1, ,±2, ±3, ±4) in the series. The other criteria is we want the terms in the Fourier series to be "small" before we cut off the Fourier series – so that the Fourier series for f(t) "looks like" f(t) and that additional terms are not making much of a difference to the sum.

At n=4, |F(n)|=0.08, which is still fairly large….this is because the f(t) function is sharp and peaky and we will need quite a few more than 4 terms to start to approximate it with the smooth sines and cosines. In fact, recalling "gibb's phenomena", we'll never exactly get there at the corners but we'll come close. If we wanted magnitude of the nth term to be less than 0.01 then we'd need at least 32 terms. For the magnitude of the nth term to get less than 0.001, then we'd need 318 terms. Actually plotting the function will give an indication of how many terms will make the Fourier series for f(t) "look like" f(t).

Potentially Useful formulae

Quadratic formula: roots of $ax^2 + bx + c$ are given by $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\sin(u) = \frac{e^{iu} - e^{-iu}}{2i} = \mathrm{Im}\left\{e^{iu}\right\} \qquad \cos(u) = \frac{e^{iu} + e^{-iu}}{2} = \mathrm{Re}\left\{e^{iu}\right\}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\,\omega_f t} \quad \text{with} \quad c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega_f t}dt \qquad \omega_f = 2\pi/T$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_f t) + b_n\sin(n\omega_f t)\right] \qquad a_0 = \frac{2}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{n2\pi t}{T}\right)dt \qquad b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{n2\pi t}{T}\right)dt$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \text{ for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \text{ for all } x$$