Solution to Midterm 2 (A)

MAT1320C, Fall 2012

1. (4 marks) Find derivatives of the following functions:

(a) $y = \arctan(\sqrt{x})$,

(b) $y = (\ln x)^x$.

Solution. (a) Let $u = \sqrt{x}$. Then $y = \arctan u$. By the chain rule,

$$y_x' = y_u' \, u_x' = \left(\frac{1}{1+u^2}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}(1+x)}.$$

(b) Use logarithmic differentiation. $\ln y = x \ln(\ln x)$.

$$y' / y = \ln(\ln x) + x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} = \ln(\ln x) + \frac{1}{\ln x}.$$

Hence, $y' = (\ln x)^x \left(\ln(\ln x) + \frac{1}{\ln x}\right) = (\ln x)^{x-1}(\ln(\ln x) \ln x + 1)$.

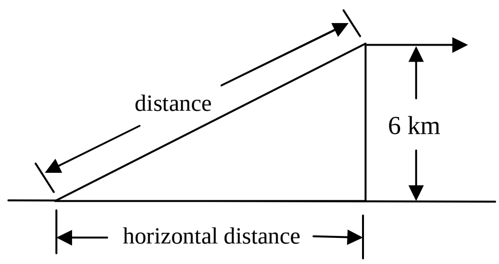
2. (3 marks) A function $y = f(x)$ is defined implicitly by the equation $x^2y + xy^2 + 2x - y = 3$.

(i) Find the derivative $f'(x)$ at the point where $x = 2$ and $y = -1$. (2 marks)
(ii) Find the equation of the tangent line of the graph of the equation at the point where $x = 2$ and $y = -1$. (1 mark)

Solution. $2xy + x^2y' + y^2 + 2xyy' + 2 - y' = 0$.

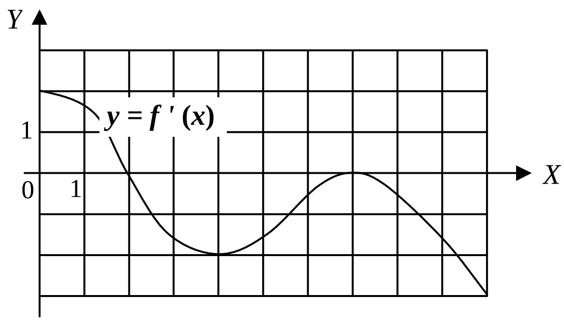
(i) When $x = 2$ and $y = -1$, $-4 + 4y' + 1 - 4y' + 2 - y' = 0$. $y' = -1$ (2 marks).
(ii) The equation of the tangent line is $y = -(x - 2) - 1$, or $y = -x + 1$ (1 mark).

3. (4 marks) A plane flies horizontally at an altitude 6 km and passes direct over a tracking telescope on the ground. When the distance between the telescope and the plane is 10 km, the distance between the telescope and the plane is increasing at a rate 10 km / min. How fast is the plane travelling (i.e., how fast is the rate of decreasing of the horizontal distance)?



Solution. Let the distance between the telescope and the plane be x and let the horizontal distance between the telescope and the plane be y. Then x and y are functions of time t, and $6^2 + y^2 = x^2$. Taking the derivative with respect to t, we have $2yy' = 2xx'$. When $x = 10$, $y = 8$. $x' = 10$, $y' = xx' / y = 100 / 8 = 12.5$ km / min. The speed of the plane is 12.5 km / min.

4. (4 marks) Suppose **the graph of the derivative** of a function $y = f(x)$ defined for $0 \le x \le 10$ is as shown in the figure below.



Answer the following questions:

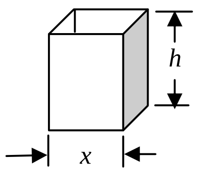
(a) For which value(s) of x, if any, does the function $f(x)$ have a local maximum? For which value(s) of x, if any, does the function $f(x)$ have a local minimum?

(b) For which value(s) of x, if any, does the graph of the function $f(x)$ have an inflection point?

Answer. (a) This function has a local maximum at $x = 2$. It does not have any local minimum.

(b) The graph of the function has inflection point at $x = 4$ and $x = 7$.

5. (5 marks) A rectangular box without the lid has a square base and its total surface area is 1200 cm^2. What is the dimension of the box so that the volume of the box is maximized and what is the maximum volume? Justify that what you got is an absolute maximum.



Solution. Let the length of the sides of the base of the box be x and let the height of the box be h. Then the volume of the box is $V = x^2h$. The total area of the box is $x^2 + 4xh = 1200$, and $h = \dfrac{1200 - x^2}{4x} = \dfrac{300}{x} - \dfrac{x}{4}$. Hence, $V = 300x - \dfrac{x^3}{4}$. Since we must have $h = \dfrac{300}{x} - \dfrac{x}{4} \ge 0$, $\dfrac{300}{x} \ge \dfrac{x}{4}$, and $x^2 \le 1200$. The domain of the function V is $0 \le x \le \sqrt{1200} = 20\sqrt{3}$. Let $V' = 300 - \dfrac{3x^2}{4} = 0$,

$x^2 = 400$, $x = 20$, $h = 10$, and $V = 4000$. Since $V' > 0$ when $x < 20$ and $V' < 0$ when $x > 20$, V attains a local maximum at $x = 20$. Since $V(0) = V(20\sqrt{3}) = 0$, this local maximum is also an absolute maximum.