Student number:__________________, Total marks: __________ out of 30

Problem	1	2	3	4
Marks				

Question 1. [8 points] The concentration of a drug in the body of a patient is reduced by 20% per day. The daily dose of this drug is d. The DTDS modeling the concentration x_t of the drug in the body on day t is

$$x_{t+1} = 0.8x_t + d. \quad = a x_t + d$$

(a) [1 point] The updating function of the DTDS is $f(x) =$ $\boxed{0.8\,x + d}$

(b) [1 point] The equilibrium point of the DTDS is $x^* =$ $\boxed{\frac{d}{0.2} = 5d}$

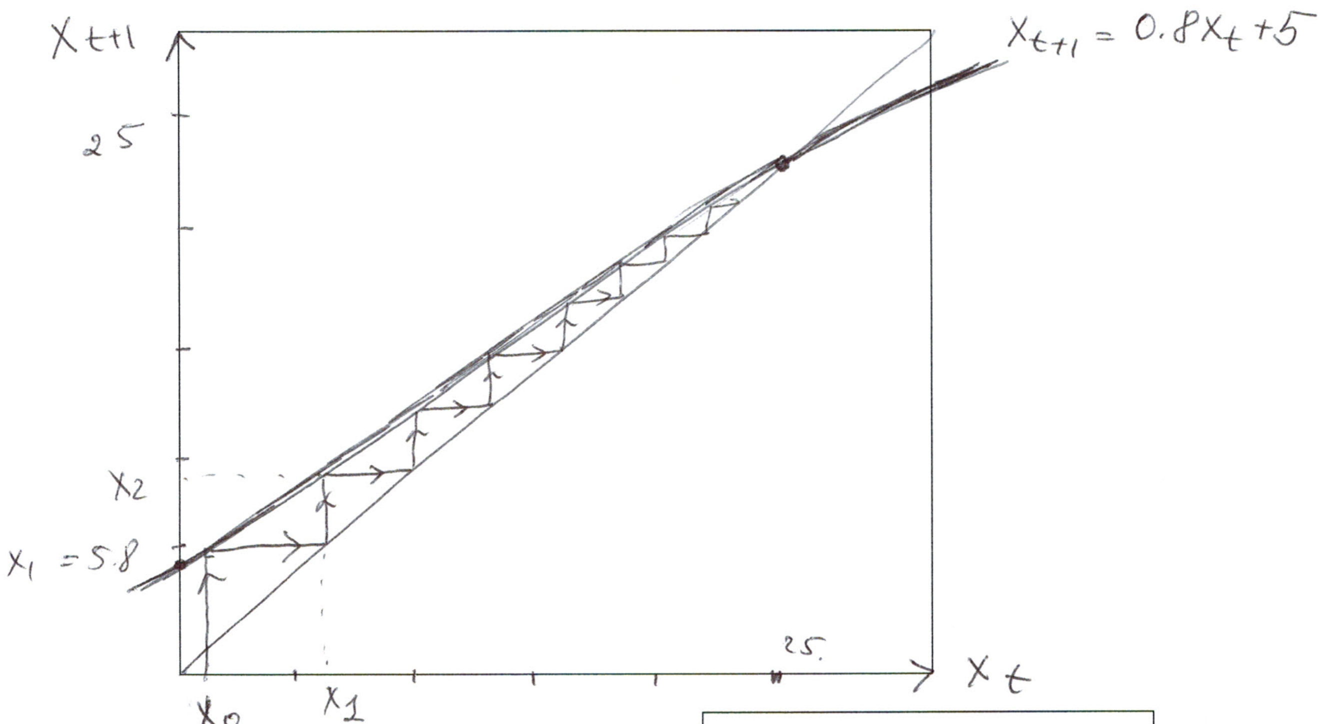
(c) [1 point] Assume the daily dose is $d = 5$. Give the solution formula for the DTDS with general initial condition x_0:

$$x_t = \boxed{(a)^t x_0 + \frac{d(1-(a)^t)}{1-a} = (0.8)^t \cdot x_0 + 5d(1-(0.8)^t)}$$

(d) [1 point] For a patient with an initial concentration of $x_0 = 3$ and a daily dose of $d = 5$, what is the concentration on day 3? $\boxed{}$

$$X_3 = (0.8)^3 \cdot 3 + 5 \cdot 5(1-(0.8)^3) = 13.736$$

(e) [2 points] Graph the updating function for $d = 5$ and draw the cobweb diagram of the DTDS, starting from $x_0 = 1$ for at least 4 steps.



(f) [1 point] Is the equilibrium point stable or unstable? Stable

(g) [1 point] Suppose that the doctors recommend a concentration of 10 in the long run. How do they have to choose the daily dose d to obtain this value? $d =$

$$X^* = \frac{d}{1-a} = 10$$

$$\frac{d}{0.2} = 10$$

$$d = 0.2 \cdot 10 = 2.$$

Question 2. [9 points] In each of the following cases, find the derivative of the function f with respect to variable x.

(a) $f(x) = \frac{1}{\sqrt{2}} e^{-(x-10)^2}$

$$f'(x) = \frac{1}{\sqrt{2}} \left[e^{-(x-10)^2} \right]' = \frac{1}{\sqrt{2}} \left[e^{-(x-10)^2} \cdot (-2(x-10)) \right]$$

(3)

(b) $f(x) = \ln\left(\frac{1}{x^2+1}\right)$

$$f'(x) = \frac{1}{\frac{1}{x^2+1}} \cdot \left(\frac{1}{x^2+1}\right)' =$$

$$= (x^2+1) \cdot \left[\frac{-2x}{(x^2+1)^2} \right] = -\frac{2x}{(x^2+1)}$$

(c) $f(x) = e^{ax}\cos(bx) + \tan^2 x$

$$f'(x) = a \cdot e^{ax} \cdot \cos(bx) - e^{ax} \cdot \sin(bx) \cdot b +$$

$$+ 2\tan x \cdot \frac{1}{\cos^2 x}$$

Question 3. [5 points] (a) Use the definition of the derivative (first principles) to calculate the derivative of the function

$$f(x) = \sqrt{x^2 - 1}.$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^2-1} - \sqrt{x^2-1}}{h} =$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{(x+h)^2-1} - \sqrt{x^2-1}\right)\left(\sqrt{(x+h)^2-1} + \sqrt{x^2-1}\right)}{h\left(\sqrt{(x+h)^2-1} + \sqrt{x^2-1}\right)} =$$

$$= \lim_{h\to 0} \left[\frac{(x+h)^2 - 1 - \cancel{\sqrt{}}\,(x^2-1)}{h\left(\sqrt{(x+h)^2-1} + \sqrt{x^2-1}\right)}\right] =$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2 + 1 - 1}{h\left(\sqrt{(x+h)^2-1} + \sqrt{x^2-1}\right)} =$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h\left(\sqrt{(x+h)^2-1} + \sqrt{x^2-1}\right)} = \lim_{h\to 0} \frac{(2x+h)}{\left(\sqrt{(x+h)^2-1} + \sqrt{x^2-1}\right)} =$$

$$= \frac{2x}{2\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}}$$

(b) Check your result, using the differentiation rules from class.

$$f'(x) = \frac{1}{2}(x^2-1)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2-1}}$$

Question 4. [8 points] Consider the function $f(x) = \frac{x-1}{x-2}$.

(a) [1 point] Find the domain of f.

$x \neq 2$

(b) [1 point] Find the limits of f as x approaches $\pm\infty$

$$\lim_{x \to +\infty} \frac{x-1}{x-2} = 1$$

$$\lim_{x \to -\infty} \frac{x-1}{x-2} = 1$$

(c) [1 point] Are there points where f is not continuous? If yes, find the left and right limit in each case.

$\underline{x = 2}$ $\qquad \to 2 \leftarrow$

$$\lim_{x \to 2^+} \frac{x-1}{x-2} = \left[\frac{1}{0^+}\right] = +\infty.$$

$$\lim_{x \to 2^-} \frac{x-1}{x-2} = \left[\frac{1}{0^-}\right] = -\infty.$$

(d) [2 point] Find the intervals where f is increasing and decreasing. Are there critical points?

$$f'(x) = \frac{(x-1)' \cdot (x-2) - (x-2)' \cdot (x-1)}{(x-2)^2} =$$

$$= \frac{(x-2) - (x-1)}{(x-2)^2} = \frac{x-2-x+1}{(x-2)^2} = -\frac{1}{(x-2)^2} = -(x-2)^{-2}$$

$f'(x) < 0$ for all $x \neq 2$. f-n is decreasing.

(e) [2 point] Find the intervals where f is concave up or concave down.

$$f''(x) = 2 \cdot (x-2)^{-3} = \frac{2}{(x-2)^3}$$

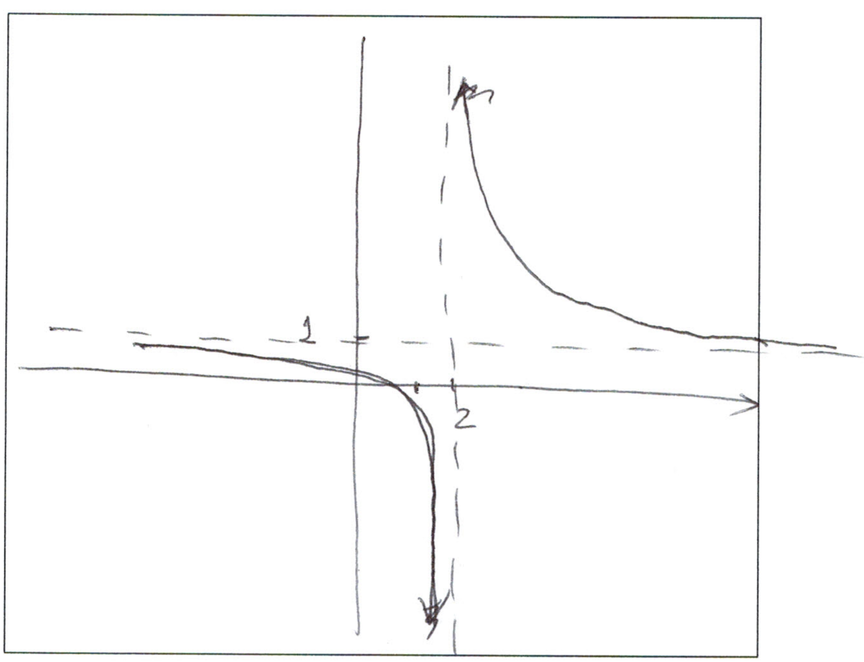
CD CU

$-$ $+$

2

f''

(f) [1 point] Draw the graph of f.



Student number:____________________, Total marks: __________ out of 30

Problem	1	2	3	4
Marks				

Question 1. [8 points] The concentration of a drug in the body of a patient is reduced by 40% per day. The daily dose of this drug is d. The DTDS modeling the concentration x_t of the drug in the body on day t is

$$x_{t+1} = 0.6x_t + d.$$

(a) [1 point] The updating function of the DTDS is $f(x) =$ $\boxed{0.6x + d}$

(b) [1 point] The equilibrium point of the DTDS is $x^* =$ $\boxed{\frac{5}{2}d \text{ or } 2.5d}$

$x^* = 0.6x^* + d$

$0.4x^* = d \Rightarrow x^* = \frac{5}{2}d$ or $x^* = 2.5d$

(c) [1 point] Assume the daily dose is $d = 5$. Give the solution formula for the DTDS with general initial condition x_0:

$x_t = (x_0 - x^*)0.6^t + x^*$

$x^* = 2.5(5) = 12.5$

$x_t =$ $\boxed{(x_0 - 12.5)(0.6)^t + 12.5}$

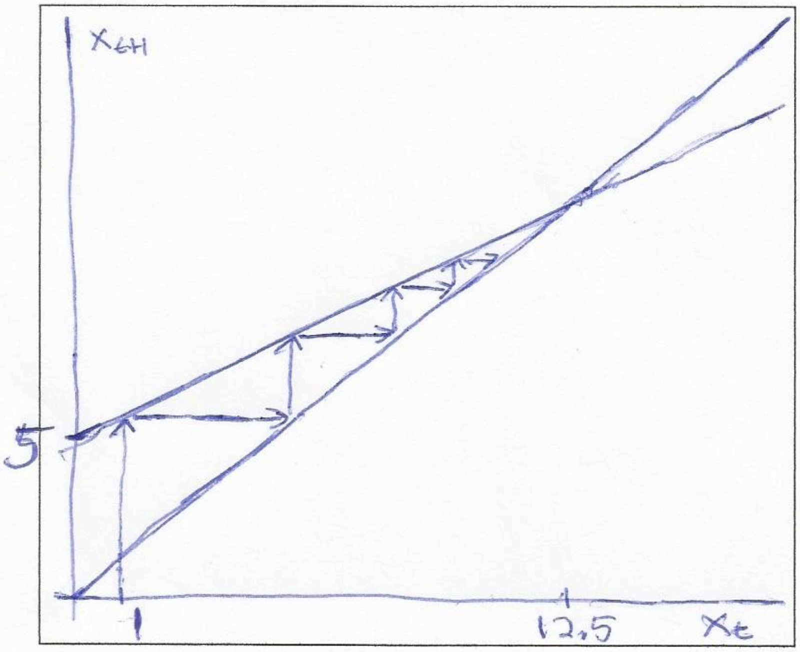
or $\boxed{x_t = (x_0 - \frac{25}{2})(0.6)^t + \frac{25}{2}}$

(d) [1 point] For a patient with an initial concentration of $x_0 = 3$ and a daily dose of $d = 5$, what is the concentration on day 3? $\boxed{10.448}$

$x_3 = (3 - 12.5)(0.6)^3 + 12.5$

$x_3 = 10.448$

(e) [2 points] Graph the updating function for $d = 5$ and draw the cobweb diagram of the DTDS, starting from $x_0 = 1$ for at least 4 steps.



(f) [1 point] Is the equilibrium point stable or unstable? stable

(g) [1 point] Suppose that the doctors recommend a concentration of 10 in the long run. How do they have to choose the daily dose d to obtain this value? $d =$ 4

$x^* = 10$

$\frac{5}{2} d = 10$

$d = 4$

Question 2. [9 points] In each of the following cases, find the derivative of the function f with respect to variable x.

(a) $f(x) = \frac{1}{\sqrt{3}} e^{-(x+10)^2}$

$$f'(x) = \frac{1}{\sqrt{3}} e^{-(x+10)^2} \cdot (-2(x+10))$$

$$= \frac{-2(x+10)}{\sqrt{3}} e^{-(x+10)^2}$$

(b) $f(x) = \ln\left(\frac{1}{x^2+3}\right)$

$$f(x) = -\ln(x^2+3)$$

$$f'(x) = -\frac{1}{x^2+3} \cdot 2x$$

$$\boxed{\frac{-2x}{x^2+3}}$$

or

$$f(x) = \ln\left(\frac{1}{x^2+3}\right)$$

$$f'(x) = \frac{1}{\frac{1}{x^2+3}} \cdot \left(-(x^2+3)^{-2}\right) \cdot 2x$$

$$= \frac{-2x(x^2+3)}{(x^2+3)^2}$$

$$= \frac{-2x}{x^2+3}$$

(c) $f(x) = e^{-ax}\sin(bx) + \tan^2 x$

$$f'(x) = -ae^{-ax}\sin(bx) + e^{-ax}\cos(bx)\cdot b + 2\tan x \cdot \sec^2 x$$

$$= e^{-ax}(-a\sin(bx) + b\cos(bx)) + 2(\tan x)(\sec^2 x)$$

Question 3. [5 points] (a) Use the definition of the derivative (first principles) to calculate the derivative of the function

$$f(x) = \sqrt{x^2+1}.$$

4 pts

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} \cdot \frac{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}$$

$$= \lim_{h\to 0} \frac{(x+h)^2+1 - (x^2+1)}{h(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})} = \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})}$$

$$= \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} = \frac{2x}{2\sqrt{x^2+1}}$$

$$= \boxed{\frac{x}{\sqrt{x^2+1}}}$$

(b) Check your result, using the differentiation rules from class.

1 pt

$$f(x) = (x^2+1)^{1/2}$$

$$f'(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x$$

$$= \frac{x}{\sqrt{x^2+1}}$$

Question 4. [8 points] Consider the function $f(x) = \frac{x-1}{2-x}$.

(a) [1 point] Find the domain of f.

$x \neq 2$

All real numbers except $x = 2$

$(-\infty, 2) \cup (2, \infty)$

(b) [1 point] Find the limits of f as x approaches $\pm\infty$

$$\lim_{x \to +\infty} \frac{x-1}{2-x} = -1$$

$$\lim_{x \to -\infty} \frac{x-1}{2-x} = -1$$

(c) [1 point] Are there points where f is not continuous? If yes, find the left and right limit in each case.

yes, f is not continuous at $x = 2$

$$\lim_{x \to 2^-} \frac{x-1}{2-x} = +\infty$$

$$\lim_{x \to 2^+} \frac{x-1}{2-x} = -\infty$$

(d) [2 point] Find the intervals where f is increasing and decreasing. Are there critical points?

$$f'(x) = \frac{(2-x) - -1(x-1)}{(2-x)^2} = \frac{1}{(2-x)^2}$$

$f'(x)$

$+ \quad +$

$2 \qquad x$

~~There are no critical points~~

f is increasing on $(-\infty, 2) \cup (2, \infty)$

f is never decreasing

(e) [2 point] Find the intervals where f is concave up or concave down.

$f'(x) = (2-x)^{-2}$

$f''(x) = -2(2-x)^{-3} \cdot -1$

$= \dfrac{2}{(2-x)^3}$

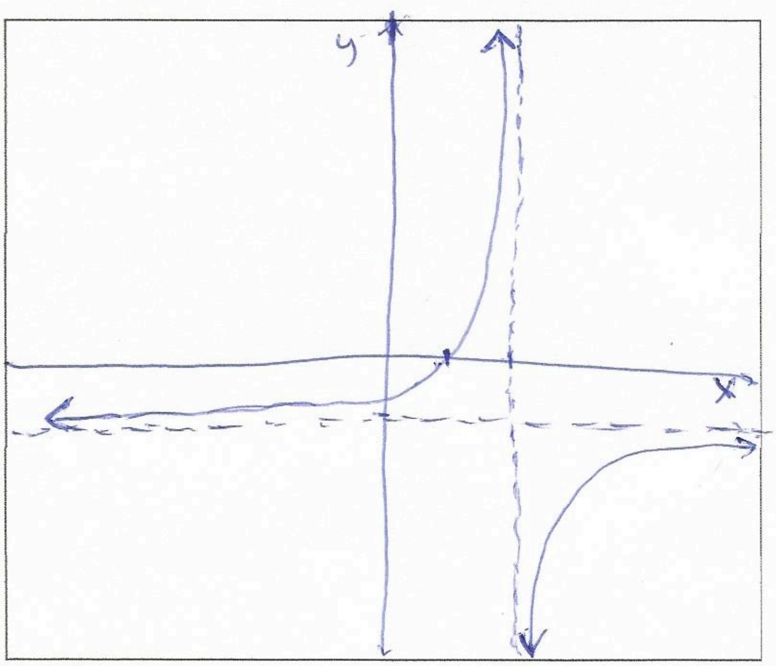
$f''(x)$

+ | −

2 x

$\boxed{\begin{array}{l} f \text{ is concave up on } (-\infty, 2) \\ f \text{ is concave down on } (2, \infty) \end{array}}$

(f) [1 point] Draw the graph of f.



Student number:__________________, Total marks: __________ out of 30

Problem	1	2	3	4
Marks				

Question 1. [8 points] The concentration of a drug in the body of a patient is reduced by 60% per day. The daily dose of this drug is d. The DTDS modeling the concentration x_t of the drug in the body on day t is

$$x_{t+1} = 0.4x_t + d.$$

(a) [1 point] The updating function of the DTDS is $f(x) =$ $\boxed{0.4x + d}$

(b) [1 point] The equilibrium point of the DTDS is $x^* =$ $\boxed{\frac{d}{1-0.4} = \frac{d}{0.6} = \frac{10d}{6}}$

(c) [1 point] Assume the daily dose is $d = 5$. Give the solution formula for the DTDS with general initial condition x_0:

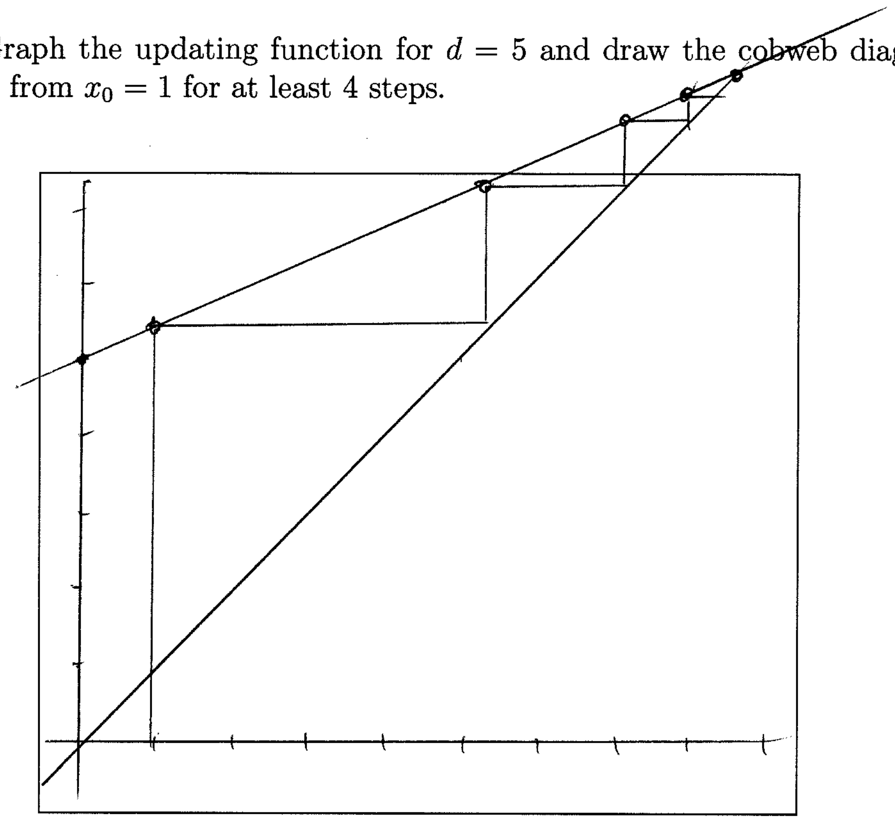
$$x_t = \boxed{(0.4)^t x_0 + 5\,\frac{1-(0.4)^t}{1-(0.4)} = (0.4)^t(x_0 - x^*) + x^*}$$

(d) [1 point] For a patient with an initial concentration of $x_0 = 3$ and a daily dose of $d = 5$, what is the concentration on day 3? $\boxed{x_3 = 7.99}$

$d = 5 \Rightarrow x^* = \frac{50}{6} \simeq 8.33$

$x_0 = 3 \Rightarrow x_3 = (0.4)^3\left(3 - \frac{50}{6}\right) + \frac{50}{6} = \frac{50}{6} - \frac{32}{6}\cdot\frac{64}{1000} = 7.99$

(e) [2 points] Graph the updating function for $d = 5$ and draw the cobweb diagram of the DTDS, starting from $x_0 = 1$ for at least 4 steps.



(f) [1 point] Is the equilibrium point stable or unstable? stable since $0.4 < 1$

(g) [1 point] Suppose that the doctors recommend a concentration of 10 in the long run. How do they have to choose the daily dose d to obtain this value? $d =$ 6

$$x^* = \frac{d}{0.6} = \frac{10d}{6} \overset{!}{=} 10$$

$$\Rightarrow d = 6$$

Question 2. [9 points] In each of the following cases, find the derivative of the function f with respect to variable x.

(a) $f(x) = \frac{1}{\sqrt{4}} e^{-(x-8)^2}$

$$f'(x) = \frac{1}{\sqrt{4}} e^{-(x-8)^2} \cdot (-2(x-8)) = \frac{16-2x}{\sqrt{4}} e^{-(x-8)^2}$$

(b) $f(x) = \ln\left(\frac{1}{x^4+2}\right) = -\ln(x^4+2)$

$$f'(x) = -\frac{4x^3}{x^4+2}$$

(c) $f(x) = e^{ax}\tan(bx) + \sin^2 x$

$$f'(x) = ae^{ax}\tan(bx) + be^{ax}\sec^2(bx) + 2\sin x\cos x$$

Question 3. [5 points] (a) Use the definition of the derivative (first principles) to calculate the derivative of the function

$$f(x) = \sqrt{x^2 - 2}.$$

$$\lim_{h \to 0} \frac{1}{h}\left[f(x+h) - f(x)\right] = \lim_{h \to 0} \frac{1}{h}\left[\sqrt{(x+h)^2 - 2} - \sqrt{x^2 - 2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\left(\sqrt{(x+h)^2-2} - \sqrt{x^2-2}\right)\left(\sqrt{(x+h)^2+2} + \sqrt{x^2-2}\right)}{\left(\sqrt{(x+h)^2-2} + \sqrt{x^2-2}\right)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x+h)^2 - 2 - (x^2 - 2)}{\sqrt{(x+h)^2-2} + \sqrt{x^2-2}}\right]$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{2xh + h^2}{\sqrt{(x+h)^2-2} + \sqrt{x^2-2}}$$

$$= \lim_{h \to 0} \frac{2x}{\sqrt{(x+h)^2-2} + \sqrt{x^2-2}} = \frac{2x}{2\sqrt{x^2-2}} = \frac{x}{\sqrt{x^2-2}}$$

(b) Check your result, using the differentiation rules from class.

$$f'(x) = \frac{2x}{2\sqrt{x^2-2}} = \frac{x}{\sqrt{x^2-2}}$$

Question 4. [8 points] Consider the function $f(x) = \frac{1-x}{x-2}$.

(a) [1 point] Find the domain of f.

$\{x \in \mathbb{R},\ x \neq 2\}$

(b) [1 point] Find the limits of f as x approaches $\pm\infty$

$$\lim_{x \to \infty} \frac{1-x}{x-2} = \lim_{x \to \infty} \frac{\frac{1}{x} - 1}{1 - \frac{2}{x}} = -1$$

$$\lim_{x \to -\infty} \frac{1-x}{x-2} = \lim_{x \to -\infty} \frac{\frac{1}{x} - 1}{1 - \frac{2}{x}} = -1$$

(c) [1 point] Are there points where f is not continuous? If yes, find the left and right limit in each case.

f is not defined at $x=2$. Continuous everywhere.

$$\lim_{x \to 2^+} \frac{1-x}{x-2} = -\infty, \quad \lim_{x \to 2^-} \frac{1-x}{x-2} = \infty$$

(d) [2 point] Find the intervals where f is increasing and decreasing. Are there critical points?

$$f'(x) = \frac{-(x-2)-(1-x)}{(x-2)^2} = \frac{1}{(x-2)^2} > 0$$

f is increasing everywhere. The only critical point is $x=2$.

(e) [2 point] Find the intervals where f is concave up or concave down.

$$f''(x) = -\frac{2(x-2)}{(x-2)^4} = -\frac{2}{(x-2)^3}$$

$f''(x) > 0$ when $x-2 < 0$ or $x < 2$ concave up

$f''(x) < 0$ when $x-2 > 0$ or $x > 2$ concave down

(f) [1 point] Draw the graph of f.