P1.1a $f(t) = t\,u(t)$

$$F(s) = \int_{0^-}^{\infty} t\,u(t)e^{-st}dt = \int_{0^-}^{\infty} t e^{-st} dt$$

use integration by parts

$$= -\frac{t}{s}e^{-st}\Big|_{0^-}^{\infty} + \int_{0^-}^{\infty} \frac{e^{-st}}{s} dt$$

$u = t$ $\quad dv = e^{-st}dt$

$du = dt$ $\quad v = -\frac{e^{-st}}{s}$

$$= \frac{e^{-st}}{s^2}(-st-1)\Big|_{0^-}^{\infty}$$

Using l'Hopital's rule $\to$ $F(s)\Big|_{t\to\infty} = \frac{-e^{-st}s}{s^3}\Big|_{t\to\infty} = 0$

$$\therefore F(s) = \frac{1}{s^2}$$

b) $f(t) = \cos\omega t\, u(t)$

$$F(s) = \int_{0^-}^{\infty} \cos\omega t\, e^{-st}dt$$

Integration by parts

$u = \cos\omega t$ $\quad dv = e^{-st}dt$

$du = -\omega\sin\omega t\,dt$ $\quad v = \frac{-e^{-st}}{s}$

$$= \frac{-e^{-st}\cos\omega t}{s}\Big|_{0^-}^{\infty} - \int \frac{\omega e^{-st}}{s}\sin\omega t\,dt\Big|_{0^-}^{\infty}$$

$$= \frac{-e^{-st}\cos\omega t}{s}\Big|_{0}^{\infty} - \frac{\omega}{s}\left[\frac{-e^{-st}}{s}\sin\omega t + \frac{\omega}{s}\int e^{-st}\cos\omega t\,dt\right]_0^{\infty}$$

Integration by parts

$u = \sin\omega t$ $\quad dv = e^{-st}dt$

$du = \omega\cos\omega t\,dt$ $\quad v = \frac{-e^{-st}}{s}$

Combining common integrals

$$\left[1 + \frac{\omega^2}{s^2}\right]\int_{0^-}^{\infty} e^{-st}\cos\omega t\,dt = \frac{+e^{-st}}{s}\left[-\cos\omega t + \frac{\omega}{s}\sin\omega t\right]_{0^-}^{\infty} = \frac{1}{s}$$

$$\therefore F(s) = \int_{0^-}^{\infty} \cos\omega t\, e^{-st}dt = \frac{\frac{1}{s}}{1 + \omega^2/s^2} = \frac{s}{s^2 + \omega^2}$$

$$y(t) = t\,[u(t)-u(t-1)] + (t+1)\,[u(t-1)-u(t-2)]$$
$$+\, 3\,[u(t-2)-u(t-4)]$$

$$= t\,u(t) + u(t-1) + (2-t)\,u(t-2) - 3\,u(t-4)$$

$$= r(t) + u(t-1) - r(t-2) - 3\,u(t-4)$$

with $r(t) = t\,u(t)$

$$\mathcal{L}\{y(t)\} = \frac{1}{s^2} + e^{-s}\frac{1}{s} - e^{-2s}\frac{1}{s^2} - 3e^{-4s}\frac{1}{s}$$

P1.3 $$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = \frac{d^3x}{dt^3} + 4\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x$$

Laplace transform (ICs = 0)

$$s^3Y(s) + 3s^2Y(s) + 5sY(s) + Y(s) = s^3X(s) + 4s^2X(s) + 6sX(s) + 8X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{s^3+4s^2+6s+8}{s^3+3s^2+5s+1}$$

P1.4 (5th edition)

a) $\ddot{x}(t) + 2\dot{x}(t) + 7x(t) = f(t)$

b) $\ddot{x}(t) + 15\dot{x}(t) + 56x(t) = 10f(t)$

c) $\dddot{x}(t) + 8\ddot{x}(t) + 9\dot{x}(t) + 15x(t) = \dot{f}(t) + 2f(t)$

P.1.4 (6th edition)

a) $\ddot{x}(t) + 5\dot{x}(t) + 10x(t) = 7f(t)$

b) $\ddot{x}(t) + 21\dot{x}(t) + 110x(t) = 15f(t)$

c) $\dddot{x}(t) + 11\ddot{x}(t) + 12\dot{x}(t) + 18x(t) = \dot{f}(t) + 3f(t)$

P1.5

a) $F_1(s) = \dfrac{5(s+2)}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1}$

$$A = \left.\frac{5(s+2)}{s+1}\right|_{s=0} = \frac{10}{1} = 10$$

$$B = \left.\frac{5(s+2)}{s}\right|_{s=-1} = \frac{5(-1+2)}{-1} = -5$$

$$f_1(t) = 10u(t) - 5e^{-t}u(t)$$

b) $f_1(0^+) = \lim_{s\to\infty} sF_1(s) = \lim_{s\to\infty} \dfrac{5(s+2)}{s+1} = 5$

c) $f_1(\infty) = \lim_{s\to 0} sF_1(s) = \lim_{s\to 0} \dfrac{5(s+2)}{s+1} = 10$

d) $f_1(0^+) = 10 - 5e^{0} = 5$

$f_1(\infty) = 10 - 5e^{-\infty} = 10$

a) $F_2(s) = \dfrac{s^2+2s+4}{s^2} = 1 + \dfrac{2}{s} + \dfrac{4}{s^2}$

$f_2(t) = \delta(t) + 2u(t) + 4r(t) = \delta(t) + 2 + 4t$ for $t \geq 0$

b) $f_2(0^+) = \lim_{s\to\infty} sF_2(s) = \lim_{s\to\infty} s + 2 + 4/s = \infty$

c) $f_2(\infty) \rightarrow sF_2(s) = \dfrac{s^2+2s+4}{s}$ No final value due to s

d) $\lim_{t\to 0^+} f_2(t) = \lim_{t\to 0^+} \delta(t) + 2 + 4t = \boxed{2}$ Do not match due to $\delta(t)$ at $t=0$

$\lim_{t\to\infty} f_2(t) = \lim_{t\to\infty} \delta(t) + 2 + 4t = \infty$

Confirms no final value

a) $F_3(s) = \dfrac{3s+7}{4s^2+24s+136} = \dfrac{3s+7}{4(s+3-5j)(s+3+5j)}$

$= \dfrac{c_1}{s+3-5j} + \dfrac{c_2}{s+3+5j}$

$c_1 = \lim_{s \to -3+5j} (s+3-5j)F_3(s) = \lim_{s \to -3+5j} \dfrac{3s+7}{4(s+3+5j)}$

$= \dfrac{-2+15j}{40j} = \dfrac{15+2j}{40}$

$= \left| \dfrac{15+2j}{40} \right| e^{j\varphi} = \dfrac{\sqrt{229}}{40} e^{j\varphi}$

where $\varphi = \tan^{-1}(2/15) = 0.1326$ rads

$c_2 = \overline{c_1}$

$\therefore\ f_3(t) = 2|c_1| e^{-3t} \cos(5t+\varphi) = \dfrac{\sqrt{229}}{20} e^{-3t} \cos(5t+0.1326)$

Alternatively, completing the square gives

$F_3(s) = \dfrac{1}{4} \dfrac{3s+7}{(s+3)^2+25} = \dfrac{1}{4}\left[\dfrac{3(s+3)}{(s+3)^2+5^2} + \dfrac{(\frac{-2}{5})5}{(s+3)^2+5^2} \right]$

$f_3(t) = \frac{3}{4} e^{-3t} \cos(5t)u(t) - \frac{1}{10} e^{-3t} \sin(5t)u(t)$

b) $f_3(0^+) = \lim_{s \to \infty} sF_3(s) = \lim_{s \to \infty} \dfrac{3s^2+7s}{4s^2+24s+136} = 3/4$

c) $f_3(\infty) = \lim_{s \to 0} sF_3(s) = 0$

d) $f_3(0^+) = \frac{3}{4} e^0 (1) - \frac{1}{10} e^0 (0) = 3/4$

$\lim_{t \to \infty} f_3(t) = \frac{3}{4}(0) - \frac{1}{10}(0) = 0$