MATH1104 Notes – By Eric Hua

Contents

Chapter 1. Geometry of $\mathbb{R}^2$ and $\mathbb{R}^3$

1.1 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Algebraic representation of vectors:

- Vectors in $\mathbb{R}^2$: $\vec{v} = \mathbf{v} = (a, b) = \begin{bmatrix} a \\ b \end{bmatrix}$, zero vector $\mathbf{0} = (0, 0)$.

- Vectors in $\mathbb{R}^3$: $\mathbf{v} = (a, b, c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, zero vector $\mathbf{0} = (0, 0, 0)$.

- Length (norm, magnitude) $||(a, b)|| = \sqrt{a^2 + b^2}$, $||(a, b, c)|| = \sqrt{a^2 + b^2 + c^2}$.

- Sum: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then $\vec{u} + \vec{v}) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$.

- Scalar multiple: Let $\vec{u} = (u_1, u_2, u_3)$, c be a scalar, then $c\vec{u} = (cu_1, cu_2, cu_3)$.

- Distance: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then $d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}||$.

- unit vector: $||\vec{u}|| = 1$.

Example 1 *Let $\vec{u} = (1, 2, -2)$. Find the unit vector which has the same direction as $\vec{u}$.*

Properties: Let c, d be scalars.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

- $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
- $\vec{u}+\vec{0}=\vec{u}$
- $\vec{u}+(-\vec{u})=\vec{0}$
- $(cd)\vec{u}=c(d\vec{u})$
- $(c+d)\vec{u}=c\vec{u}+d\vec{u}$
- $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$
- $1\vec{u}=\vec{u}$
- $(-1)\vec{u}=-\vec{u}$
- $0\vec{u}=\vec{0}$
- $\vec{u}//\vec{v} \Leftrightarrow \vec{v}=c\vec{u}$

1.2 Dot and cross product

In this section, we study two operations.

- Dot product: Let $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$, then $\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2+u_3v_3$.

- Angle: Let θ be the angle between $\vec{u}$ and $\vec{v}$ which satisfies $0\leq\theta\leq\pi$, then $\cos\theta=\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}$.

- Orthogonal: $\vec{u}\perp\vec{v}$ if $\vec{u}\cdot\vec{v}=0$.

Example 2 *Let $\vec{u}=(1,2,-2)$, $\vec{v}=(-2,-2,1)$, then $\vec{u}\cdot\vec{v}=-8$.*

Example 3 *Let $\vec{u}=(1,2,-2)$, $\vec{v}=(-2,-2,1)$, Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.*

Solution:
$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||} = \frac{-8}{9}.$$
Properties: Let c be a scalar.

- $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
- $\vec{w}\cdot(\vec{u}+\vec{v}) = \vec{w}\cdot\vec{u} + \vec{w}\cdot\vec{v}$
- $c(\vec{u}\cdot\vec{v}) = (c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v})$
- $\vec{u}\cdot\vec{0} = 0$
- $\vec{u}\cdot\vec{u} = ||\vec{u}||^2$.

- Cross product: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then $\vec{u}\times\vec{v} = (u_2v_3 - u_3v_2, +u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$.

- Orthogonal: $\vec{u}\times\vec{v}\perp\vec{u}$, $\vec{u}\times\vec{v}\perp\vec{v}$.

Example 4 *Find a vector that is orthogonal to both* $\vec{u} = (1, 2, -1)$*,* $\vec{v} = (0, 2, 3)$*.*

Solution: Any scalar multiple of $\vec{u}\times\vec{v} = (8, -3, 2)$.
Properties: Let c be a scalar.

- $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$
- $\vec{w}\times(\vec{u}+\vec{v}) = \vec{w}\times\vec{u} + \vec{w}\times\vec{v}$
- $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$
- $c(\vec{u}\times\vec{v}) = (c\vec{u})\times\vec{v} = \vec{u}\times(c\vec{v})$
- $\vec{u}\times\vec{0} = \vec{0}$
- $\vec{u}\times\vec{u} = \vec{0}$
- $||\vec{u}\times\vec{v}|| = ||\vec{u}||\,||\vec{u}||\sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$
- $||\vec{u}\times\vec{v}||$ is the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

Example 5 *Find the area of the parallelogram determined by* $\vec{u} = (1, 2, -1)$, $\vec{v} = (0, 2, 3)$.

Solution: $A = ||\vec{u} \times \vec{v}|| = ||(8, -3, 2)|| = \sqrt{77}$.

Example 6 *Find the area of the triangle with vertices* $P(1, 2, 3)$, $Q(-3, 2, 1)$, *and* $R(2, 4, 5)$.

Solution: $\vec{PQ} = Q - P = (-4, 0, -2)$, $\vec{PR} = R - P = (1, 2, 2)$.

$$A = \frac{1}{2}||\vec{PQ} \times \vec{PR}|| = ||(4, 6, -8)|| = \sqrt{29}.$$

1.3 Lines and planes

Line: A line is determined by a point and a vector (direction vector) parallel to the line. Let $P(p_1, p_2, p_3)$ be a point on the line L. Let $\vec{v}$ be a nonzero vector which is parallel L.

- Point-parallel form (vector form): $\vec{x}(t) = \vec{p} + t\vec{v}$, $t \in \mathbb{R}$, $\vec{p} = (p_1, p_2, p_3)$, $\vec{x}(t) = (x, y, z)$.
- Parametric form: $x = p_1 + tv_1$, $y = p_2 + tv_2$, $z = p_3 + tv_3$.
- Symmetric form: $\frac{x-p_1}{v_1} = \frac{y-p_2}{v_2} = \frac{z-p_3}{v_3}$.

Remark. (Two-point form). If a line goes through two points P and Q, then $\vec{x}(t) = \vec{p} + t(\vec{q} - \vec{p})$, where $\vec{q}$, $\vec{p}$ are the position vectors of Q, P.

Example 7 *Find the equation of the line through* $P(1, 2, 3)$ *and* $Q(3, 1, 1)$.

Two lines L_1 and L_2 can be

- parallel
- intersected

- skewed

Example 8 *Show that the intersection between* $L_1 : x = 1 + t, y = -2 + 3t, z = 4 - t$ *and* $L_2 : x = 2s, y = 3 + s, z = -3 + 3s$ *is* $(16/5, 23/5, 9/5)$.

Plane: A plane Π is determined by a point and a normal vector $\vec{n}$ which is perpendicular to the plane. Let $P(p_1, p_2, p_3)$ be a point on the plane. Let $\vec{n}$ be a nonzero vector which is perpendicular to the plane.

- Point-normal form: $(\vec{x} - \vec{p}) \cdot \vec{n} = 0$, $\vec{x} = (x, y, z)$.
- Parametric form: $n_1(x - p_1) + n_2(y - p_2) + n_3(zp_3) = 0$, where $(n_1, n_2, n_3) = \vec{n}$.
- Standard form: $n_1x + n_2y + n_3z = d$, where $d = n_1p_1 + n_2p_2 + n_3p_3$.

Example 9 *Find the equation of the plane through three points* $P(1, 2, 3)$*,* $Q(-3, 2, 1)$*, and* $R(2, 4, 5)$.

Solution: $\vec{PQ} = Q - P = (-4, 0, -2)$, $\vec{PR} = R - P = (1, 2, 2)$. $\vec{n} = \vec{PQ} \times \vec{PR} = (4, 6, -8)$. Thus

$$4(x - 1) + 6(y - 2) - 8(z - 3) = 0, \Rightarrow 4x + 6y - 8z = -8.$$

Distance between a point and a plane: A plane Π is determined by a point and a normal vector $\vec{n}$ which is perpendicular to the plane. Let $P(p_1, p_2, p_3)$ be a point, let Π be: $ax + by + cz = d$. Then the distance between them is:

$$D = \frac{|ap_1 + bp_2 + cp_3 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example 10 *Show that the distance between* $P(1, 2, 3)$ *and the plane* $2x - 2y - z = 1$ *is 2.*

Chapter 2. Euclidean m-Space and Linear Equations

2.1 Euclidean m-Space

Vectors in $\mathbb{R}^m$:

$$\mathbb{R}^m = \{(u_1, u_2, ..., u_m) | u_i \in \mathbb{R}\} = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} | u_i \in \mathbb{R} \right\}.$$

- Distance $d(\vec{u}, \vec{v}) = \sqrt{(u_1 - v_1)^2 + \cdots + (u_m - v_m)^2}$.
- Length (norm, magnitude) $||(u_1, u_2, ..., u_m)|| = \sqrt{u_1^2 + \cdots + u_m^2}$.
- Sum: Let $\vec{u} = (u_1, ..., u_m)$, $\vec{v} = (v_1, ..., v_m)$, then $\vec{u} + \vec{v}) = (u_1 + v_1, ..., u_m + v_m)$.
- Scalar multiple: Let c be a scalar, then $c\vec{u} = (cu_1, cu_2, ..., cu_m)$.

Properties: Let c, d be scalars.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $(cd)\vec{u} = c(d\vec{u})$

- $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
- $1\vec{u} = \vec{u}$
- $(-1)\vec{u} = -\vec{u}$
- $0\vec{u} = \vec{0}$
- $\vec{u} // \vec{v} \Leftrightarrow \vec{v} = c\vec{u}$

Dot product

- Dot product: Let $\vec{u} = (u_1, ..., u_m)$, $\vec{v} = (v_1, .., v_m)$, then $\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_m v_m$.

- Orthogonal: $\vec{u} \perp \vec{v}$ if $\vec{u} \cdot \vec{v} = 0$.

Example 11 *Let* $\vec{u} = (1, 0, 0, 0)$, $\vec{v} = (0, 1, 0, 0)$, *then* $\vec{u} \cdot \vec{v} = 0$.

Line: If a line goes through two points P and Q, then $\vec{x}(t) = (1-t)\vec{p} + t\vec{q}$, where $\vec{q}$, $\vec{p}$ are the position vectors of Q, P, $-\infty < t < \infty$.

- If $0 \le t \le 1$, then it is the line segment from P to Q.

Example 12 *Find the equation of the line through* $P(1, 2, 3, 4)$ *and* $Q(2, 3, 1, 1)$.

Solution: $x_1 = 1 + t, x_2 = 2 + t, x_3 = 3 - 2t, x_4 = 4 - 3t$.
Hyperplane: Let P be a point on the plane. Let $\vec{n}$ be a nonzero vector which is perpendicular to the plane.

- Point-normal form: $(\vec{x} - \vec{p}) \cdot \vec{n} = 0$, $\vec{x} = (x_1, x_2, ..., x_m)$.

- A plane through the point P and parallel two noncolinear vectors $\vec{u}$ and $\vec{v}$ is given by: $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$.

- Standard form (linear equation): $a_1x_1+a_2x_2+\cdots+a_mx_m = b$, where $\vec{n} = (a_1, a_2, ..., a_m)$.

Example 13 *Find the equation of the plane through three points P, Q, and R.*

Solution: $\vec{x} = \vec{p} + s(\vec{q} - \vec{p}) + t(\vec{r} - \vec{p})$.

2.2 Systems of Linear Equations

Definition 1 *A* **linear equation** *in variables $x_1, x_2, \ldots, x_n$ has the form*

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where the numbers $a_1, \ldots, a_n \in \mathbb{R}$ are the equation's coefficients and $d \in \mathbb{R}$ is the constant. An n-tuple $(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ is a **solution** *of, or satisfies, that equation if substituting the numbers s_1, ..., s_n for the variables gives a true statement: $a_1s_1 + a_2s_2 + \ldots + a_ns_n = d$.*

A **system of linear equations**

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m \end{array}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that n-tuple is a solution of all of the equations in the system.

Finding the set of all solutions is solving the system.

Example 14 *The ordered pair $(-1, 5)$ is a solution of this system.*

$$\begin{array}{ccccc} 3x_1 & + & 2x_2 & = & 7 \\ -x_1 & + & x_2 & = & 6 \end{array}$$

In contrast, (5,-1) is not a solution.

Definition 2 *If we have two linear systems and they have the same solution set then the two linear systems are called* **equivalent**.

A linear system is called inconsistent if it has no solution. Otherwise it is called consistent.

Example 15 *The system*

$$\begin{array}{rcrcl} 3x_1 & + & 2x_2 & = & 7 \\ -x_1 & + & x_2 & = & 6 \end{array}$$

has only one solution (-1,5).

Example 16 *The system*

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 7 \\ -2x_1 & - & 4x_2 & = & -14 \end{array}$$

has infinite solutions (7-2k, k).

Example 17 *The system*

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 7 \\ -2x_1 & - & hx_2 & = & k \end{array}$$

has no solution when $h = 4$ *and* $k \neq -14$*; one solution when* $h \neq 4$*; infinite solutions when* $h = 4$ *and* $k = -14$*.*

Elementary operations: There are three types of elementary operations to a system of linear equations.

1. Replacement: Replace an equation by the sum of itself and the multiple of another equation.
2. Interchange: Interchange two equations.
3. Scaling: Multiply an equation by a non-zero constant.

Theorem 1 *Elementary operations will result in an equivalent system.*

2.3 Row reduction of linear systems

Matrices

Definition 3 *An $m \times n$ (m by n)* matrix *A with m rows and n columns with entries in $\mathbb{R}$ is a rectangular array of the form*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where $\forall (i,j) \in \{1,2,\ldots,m\} \times \{1,2,\ldots,n\}, \quad a_{ij} \in \mathbb{R}$.

As a shortcut, we often use the notation $A = [a_{ij}]$ to denote the matrix A with entries a_{ij}. Notice that when we refer to the matrix we put parentheses—as in "$[a_{ij}]$," and when we refer to a specific entry we do not use the surrounding parentheses—as in "a_{ij}."

Example 18

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

is a 2×3 matrix and

$$B = \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$$

is a 3×2 matrix.

To solving linear systems, we put all the coefficients of each variable aligned in columns to get the **coefficient matrix**. By adding an additional column to the coefficient matrix consisting of the values on the right hand side of the equal sign to give the **augmented matrix**.

Example 19 *Consider this linear system*

$$\begin{array}{rcrcrcl} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

$$\text{coefficient matrix} = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 5 & -2 \\ 1/3 & 2 & 0 \end{bmatrix}, \quad \text{augmented matrix} = \begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ 1/3 & 2 & 0 & 3 \end{bmatrix}.$$

Elementary row operations: There are three types of elementary row operations.

1. Replacement: Replace one row by the sum of itself and the multiple of another: $R_i \to R_i + cR_j$.
2. Interchange: Interchange two rows: $R_i \leftrightarrow R_j$.
3. Scaling: Multiply all entries in a row by a non zero constant: $R_i \to cR_i$.

Definition 4 *Two matrices are row equivalent if one matrix can be transformed into another matrix by a sequence of elementary row operations.*

A **leading entry of a row**: is the leftmost, nonzero entry in the row (nonzero row).

Definition 5 *A rectangular matrix is in* **echelon form** *(row echelon form) if it has the following three properties.*

1. All non zero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zero.

Definition 6 *Given a matrix in echelon form, if it satisfies the following two conditions then it is in reduced echelon form (***row-reduced echelon form***)*

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

Example 20 *Classify row echelon form, reduced row echelon form, or not in echelon form:*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, E = \begin{bmatrix} 2 & 3 & 4 & 0 & 1 \\ 0 & 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, G = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We are interested in performing row operations until one of these two matrix structures arises.

Uniqueness of the Reduced Echelon Form: Each matrix is row equivalent to one and only one reduced echelon matrix.

Example 21 *Look at the following matrix:*

$$\begin{bmatrix} 1 & h-1 & 1 & 0 & 1 \\ 0 & h & 1 & 0 & 1 \\ 0 & 0 & h-1 & 1 & 0 \end{bmatrix}$$

1) Find h such that the matrix is in reduced row echelon form; ($h = 1$)

2) Find h such that the matrix is in echelon form but not in reduced row echelon form; ($h \neq 1$ and $h \neq 0$)

3) Find h such that the matrix is not in row echelon form. ($h = 0$.)

Example 22 *Carry the following matrix to (1) Echelon form; (2) reduced echelon form:*

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & 1 \\ 3 & -7 & 8 & -5 & 8 & 2 \\ 3 & -9 & 12 & -9 & 6 & 6 \end{bmatrix}.$$

Properties:

- The augmented matrix of a consistent linear system is row equivalent to a matrix with the last non-zero row
$$\begin{bmatrix} \cdots & * & * \end{bmatrix}.$$
- The augmented matrix of a inconsistent linear system is row equivalent to a matrix with the last non-zero row
$$\begin{bmatrix} 0 & \cdots & 0 & * \end{bmatrix}.$$

Gauss-Jordan elimination:

1. Construct the augmented matrix;
2. Carry the matrix to the row-reduced echelon form;
3. Change back to a system of equations;
4. Solve each equation for the unknown whose coefficient is a leading 1 (other unknowns are replaced by parameters).

Example 23 *Solve the following system by using elementary row operations:*

$$\begin{array}{ccccccc} x & + & y & - & 2z & = & -2 \\ & & y & + & 3z & = & 7 \\ x & & & - & z & = & -1 \end{array}$$

We can start by going to echelon form:

$$\text{augmented matrix } = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 1 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 4 & 8 \end{bmatrix} \xrightarrow{\frac{1}{4}R_3} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 + 2R_3, R_2 - 3R_3} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

The answer is $x = 1$, $y = 1$, and $z = 2$.

Example 24 *Solve the following system by using elementary row operations:*

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 4 \\ & & & 2x_2 & 2x_3 & = & 8 \end{array}$$

Solution: The general solution is

$$\begin{aligned} x_1 &= -7 - x_3 \\ x_2 &= 4 - x_3 \\ x_3 &= free \end{aligned}$$

Remark: x_1, x_2 here are basic variables, x_3 is called a free variable.

Remark: For a consistent system, if it has free variables, it has infinite solutions; if no free variable, then only one solution.

Chapter 3. Matrices

3.1 Operations on matrices

- Diagonal matrix: Except for entries on diagonal (main diagonal), all other entries are 0,
- Zero matrix: all entries are 0,
- Identity matrix I_n: all entries on diagonal are 1, other entries are 0.

Scalar multiplication and addition of matrices: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices, r be a number. Then

$$rA = [ra_{ij}], \quad A + B = [a_{ij} + b_{ij}].$$

Remark. You can only add matrices of the same size. Also, two matrices are equal if they are the same size and corresponding entries are equal.

Example 25

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 0+4 & -1+5 & 1+6 \\ 1+7 & 2+8 & 3+9 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 7 \\ 8 & 10 & 12 \end{bmatrix}.$$

$$3 \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & 6 \\ 0 & 9 \end{bmatrix}.$$

Properties:

Let A, B, C be matrices of the same size and let r and s be scalars.

1. A+B = B+A

2. (A+B)+C = A+(B+C)

3. A+0 = A

4. r(A+B) = rA + rB

5. (r+s)A = rA + sA

6. r(sA) = (rs)A

Matrix multiplication: Let $A = [a_{ij}]_{m\times r}$ and $B = [b_{ij}]_{r\times n}$. Then

$$AB = [c_{ij}]_{m\times n},$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} = \left[\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in} \end{array}\right] \left[\begin{array}{c} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{array}\right].$$

Example 26

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 1(7)+2(9)+3(11) & 1(8)+2(10)+3(12) \\ 4(7)+5(9)+6(11) & 4(8)+5(10)+6(12) \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Remark. In order to have the product AB of two matrices A and B, the number of columns of A must equal the number of rows of B. So, if A is an $m \times r$ and B is an $s \times n$ matrix, in order to have the product AB, we need r = s. The resulting matrix AB will be an $m \times n$ matrix.

Properties of matrix multiplication: Let A, B, C be matrices for which sums and products are defined.

1. A(BC) = (AB)C (associativity)

2. A(B+C)= AB + AC (Left distributivity)

3. (B+C)A = BA + CA (Right distributivity)

4. r(AB) = (rA)B = A(rB)

5. $I_m A = A = AI_n$, here A is m x n.

6. In general, $AB \neq BA$.

Transpose of a matrix: Given a matrix A then transpose of A is a matrix denoted by A^T, whose rows are the columns of A and whose columns are the rows of A.

Example 27

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties of Transpose:

1. $(A^T)^T = A$

2. $(A+B)^T = A^T + B^T$

3. $(rA)^T = rA^T$ where r is a scalar.

4. $(AB)^T = B^T A^T$.

3.2 Matrix equations and inverse

$A\vec{x} = \vec{b}$ is called matrix equation.

Definition 7 *Given an $n \times n$ matrix A, the inverse of A is an $n \times n$ matrix B such that*

$$BA = AB = I,$$

where I is the $n \times n$ identity. The inverse of A is denoted by A^{-1}. If A is invertible, then we say that A is nonsingular.

Example 28 *The inverse of 2×2 matrix: If $ad - bc \neq 0$, then*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Theorem 2 *If A is an invertible $n \times n$ matrix, then for each $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.*

Properties of inverses:

1. If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$

2. If A and B are invertible then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

3. If A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Theorem 3 *An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n and in this case, any sequence of elementary row operations that reduce A to I_n also transforms I_n into A^{-1}.*

Example 29 *Find A^{-1}, where*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{bmatrix}.$$

Solution:

$$[A|I] = \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 2 & 5 & 6 & | & 0 & 1 & 0 \\ 3 & 6 & 10 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1, R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 3 & | & 5 & -2 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -3 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_3} \begin{bmatrix} 1 & 0 & 0 & | & 14 & -2 & -3 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -3 & 0 & 1 \end{bmatrix}.$$

Thus

$$A^{-1} = \begin{bmatrix} 14 & -2 & -3 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Example 30 *Solve the system of equations: $A\vec{x} = \vec{b}$, where* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$

Solution: $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 14 & -2 & -3 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 17 \\ -2 \\ -4 \end{bmatrix}.$

3.3 Theory of linear systems

Theorem 4 *Any linear system has,*

1. no solution

2. one solution

3. infinitely many solutions.

Example 31 *Consider the following system of linear equations*

$$\begin{aligned} x_1 + 4x_2 - 8x_3 &= 0 \\ 2x_1 + 5x_2 - 7x_3 &= 0 \\ -3x_1 - 7x_2 + kx_3 &= 0 \end{aligned}$$

(i) Find value(s) of k such that the system has only trivial solution.

(ii) Find value(s) of k such that the system has non-trivial solutions.

(iii) For the value(s) of k in (ii), describe the solution set.

Definition 8 *A linear system $A\vec{x} = \vec{0}$ is called homogeneous. Linear system $A\vec{x} = \vec{b}$ is called non-homogeneous if $\vec{b} \neq \vec{0}$. The zero vector is a solution of any homogeneous system, it is called* **trivial solution**, *other solutions are called non-trivial solutions.*

Theorem 5 *Every homogeneous system $A\vec{x} = \vec{0}$ has either 1 or infinite solutions.*

Example 32 *Consider the following system of linear equations*

$$\begin{aligned} x_1 + 4x_2 - 8x_3 &= 0 \\ 2x_1 + 5x_2 - 7x_3 &= 0 \\ -3x_1 - 7x_2 + kx_3 &= c - 1 \end{aligned}$$

(i) Find value(s) of k and c such that the system has no solution.

(ii) Find value(s) of k and c such that the system has only one solution.

(iii) For the value(s) of k and c such that the system has infinitely many solutions.

Solution: (i)

$$\text{augmented matrix} = \begin{bmatrix} 1 & 4 & -8 & 0 \\ 2 & 5 & -7 & 0 \\ -3 & -7 & k & c-1 \end{bmatrix} \xrightarrow{R_2 - 2R_1, R_3 + 3R_1} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & k-24 & c-1 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 5 & k-24 & c-1 \end{bmatrix} \xrightarrow{R_3 - 5R_2} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & k-9 & c-1 \end{bmatrix}$$

(i) If $k = 9$, $c \neq 1$, then no solution.

(ii) For $k \neq 9$, the system has only one solution.

(iii) When $k = 9$, $c = 1$, infinitely many solutions.

Definition 9 *The rank of a matrix A = the number of leading entries in echelon form = the number of leading 1s in the row-reduced echelon form = the number of nonzero rows in the row-reduced echelon form.*

Example 33 *Let* $A = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 4 & 7 & 4 \\ 0 & 0 & 8 & 14 & 8 \end{bmatrix}$. *Then the rank=2.*

Theorem 6 *For a linear system $A\vec{x} = \vec{b}$, where A is $m \times n$, let p be the rank of the coefficient matrix, let q be the rank of the augmented matrix. Then the system has*

1. *no solution, if $p < q$;*
2. *only one solution if $p = q = n$;*
3. *infinitely many solutions if $p = q < n$.*

Corollary:

1. $A\vec{x} = \vec{0}$ has only trivial solution if $rank(A) =\#$ of columns of A.
2. $A\vec{x} = \vec{0}$ has infinitely many solutions if $rank(A) <\#$ of columns of A (in particular, more unknowns than equations).

Theorem 7 *If $\vec{p}$ is a solution of $A\vec{x} = \vec{b}$, and $\vec{v_h}$ is a solution of $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{v_h}$ is a solution of $A\vec{x} = \vec{b}$*

Example 34 *Let* $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 8 & 6 \\ 5 & -13 & -8 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$. *Describe all solutions of $A\vec{x} = \vec{b}$.*

Sol:

$$[A|\vec{b}] = \begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 8 & 6 & -2 \\ 5 & -13 & -8 & 3 \end{bmatrix} \xrightarrow{R_2 + 3R_1, R_3 - 5R_1} \begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & -1 & -6 & 1 \\ 0 & 2 & 12 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & -1 & -6 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 14 & 4 \\ 0 & -1 & -6 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 0 & 14 & 4 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 14x_3 \\ 1 - 6x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -14x_3 \\ -6x_3 \\ x_3 \end{bmatrix} + = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -14 \\ -6 \\ 1 \end{bmatrix}.$$

Theorem 8 *(The Invertible Matrix Theorem) Let A be a square $n \times n$ matrix. Then the following statements are equivalent.*

1. *A is an invertible matrix.*
2. *The equation $A\vec{x} = \vec{b}$ has a unique solution for each $\vec{b} \in \mathbb{R}^n$.*
3. *The equation $A\vec{x} = \vec{0}$ has only the trivial solution.*
4. *A is row equivalent to the identity matrix.*
5. *The rank(A)=n.*

3.5 Elementary matrices and linear systems

Elementary Matrices:

An elementary matrix is a matrix obtained by performing one elementary row operation onto an identity matrix. Every elementary matrix is invertible. The inverse of an elementary matrix E is again an elementary matrix.

- $E_{ij}(c)$ is the matrix obtained from I by replacing its (i, j)-entry by c. $E_{ij}(c)^{-1} = E_{ij}(-c)$.
- P_{ij} is the matrix obtained from I by interchanging rows i and j. $P_{ij}^{-1} = P_{ij}$.

Theorem 9
- *$E_{ij}(c)A$ is equivalent to the elementary row operation $R_i \to R_i + cR_j$ to A.*
- *$E_{ii}(c)A$ is equivalent to the elementary row operation $R_i \to cR_i$ to A.*
- *$P_{ij}A$ is equivalent to the elementary row operation $R_i \leftrightarrow R_j$ to A.*

LU decomposition: If

$$E_n \cdots E_2 E_1 A = U,$$

where E_i are elementary matrices and U is in echelon form, then

$$A = LU, \quad L = E_1^{-1} E_2^{-1} \cdots E_n^{-1}.$$

Example 35 *Let* $A = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 1 & 0 & -1 & -1 \end{bmatrix}$. *Find LU decomposition.*

Solution: We can start by going to echelon form:

$$A = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 1 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 4 & 8 \end{bmatrix} = U.$$

Thus

$$E_{32}(1)E_{31}(-1)A = U, \Rightarrow L = E_{31}(-1)^{-1}E_{32}(1)^{-1} = E_{31}(1)E_{32}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

3.6 Application–Least square polynomials

Definition 10 *Given the points* $(x_0, y_0), (x_1, y_1), \cdots, (x_m, y_m)$*, if the polynomial*

$$Q(x) = b_0 + b_1 x + \cdots + b_n x^n, \quad n \leq m$$

minimizes

$$S = [Q(x_1) - y_1]^2 + \cdots + [Q(x_m) - y_m]^2,$$

then $Q(x)$ *is called the nth-degree* **least square polynomial** *for these points.*

Theorem 10 *The coefficients of* $Q(x)$ *satisfy the following equation (it is called* **normal equation***):*

$$(U^T U)\vec{x} = U^T \vec{y}, \quad U = \begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & \cdots & x_m^n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

Example 36 *Find the least squares approximating line for the data points (1,2), (2,2) and (3,4).*

Solution: Let the line be $y = a + bx$. Then

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$$

The normal equation is:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}^T \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \Rightarrow \begin{bmatrix} 3 & 6 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 18 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}, \Rightarrow y = 1.5 + x.$$

Example 37 *Find the least squares approximating parabola for the data points (-1,1), (0,-1) and (1,0), and (2,2).*

Solution: Let the parabola be $y = Q(x) = a + bx + cx^2$. Then

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$

The normal equation is:

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}, \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -0.7 \\ -0.6 \\ 1 \end{bmatrix}.$$

$$Q(x) = -0.7 - 0.6x + x^2.$$

Chapter 4. Determinants

4.1 Definition of a determinant

Definition 11 *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *The determinant of* A *is defined as*

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For a $n \times n$ matrix A, let A_{ij} be the matrix obtained from A by deleting the i-th row and j-th column. The $(i, j)^{th}$ cofactor of A is the number,

$$c_{ij} = (-1)^{i+j} \det A_{ij}.$$

$$\det A = a_{i1}c_{i1} + a_{i2}c_{i2} + ... + a_{in}c_{in},$$

which is called a cofactor expansion across the i-th row. Similarly,

$$\det A = a_{1j}c_{1j} + a_{2j}c_{2j} + ... + a_{nj}c_{nj},$$

which is called a cofactor expansion across the j-th column.

Example 38 *Calculate* $\det A$*, where*

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}.$$

Solution: We do cofactor expansion across the 2nd row.

$$\begin{aligned} \det A &= a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23} \\ &= 2(-1)^{2+1} \det \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} + (-1)^{2+2} \det \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} + (-1)^{2+3} \det \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \\ &= 2(14) + (-13) + 5 = 20. \end{aligned}$$

Definition 12 *A triangular matrix is a matrix that is all zeros either above or below the diagonal. An upper triangular matrix means all entries below the main diagonal are zero; an lower triangular matrix means all entries above the main diagonal are zero.*

Theorem 11 *If A is a triangular matrix then $detA$ is the product of the entries on the main diagonal of A.*

Example 39 *Calculate* $\det A$*, where*

$$A = \begin{bmatrix} 5 & 3 & 5 & 7 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 12 \end{bmatrix}.$$

Solution: A is an upper triangular matrix. $\det A = 5(1)(2)(12) = 120$.

Properties:

- If a square matrix has a zero row or column, then its determinant is 0.
- If a square matrix has two same rows or two same columns, then its determinant is 0.
- If A is a 3×3 matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

  Remark. This comes from the three main diagonals and three other diagonals by repeating the first two columns.

4.2 Properties of Determinants

Theorem 12 *(Row Operations)*

1. If a multiple of one row of A is added to another row to produce a matrix B then $\det B = \det A$*.*

2. If two rows of A are interchanged to produce B then detB = -detA.

3. If one row of A is multiplied by k to produce B then det B = k det A.

Example 40 *Let* $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$*. Find* $\det(A)$*.*

Solution:

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 - 3R_3} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

If we denote the above three new matrices as B, C, D, then

$$\det A = \det B = \det C = -\det D = -(1)(-2)(-1) = -2.$$

Properties of determinants:

1. If A is a square matrix then $\det A^T = \det A$.

2. $\det(AB) = \det A \det B$.

3. If A is an $n \times n$ matrix and c a scalar then $\det(cA) = c^n \det A$.

4. If A is invertible then $det(A^{-1}) = \frac{1}{\det A}$.

5. A square matrix A is invertible $\Leftrightarrow \det A \neq 0$.

Example 41 *Let A, B and C be* 3×3 *invertible matrices,* $det(A) = 3$, $det(B) = 5$, $det(C) = 6$. *Calculate* $det(A^{-1}C(-2B))$.

Solution.

$$det(A^{-1}C(-2B)) = det(A^{-1})det(C)(-2)^3 det(B) = -8\frac{1}{det(A)}det(C)det(B)$$

$$= -8\frac{1}{3}(6)5 = -80.$$

Example 42 *(Vandermonde determinant)*

$$det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} = \prod_{i>j;\ i,j=1}^{n} (x_i - x_j).$$

4.3 Cramer's rule

Suppose we have a vector equation $A\vec{x} = \vec{b}$. Cramers Rule is a way of determining the solution $\vec{x}$ without creating an augmented matrix and performing elementary row operations on it.

Cramers Rule: Let A be an invertible $n \times n$ matrix. For any $\vec{b} \in \mathbb{R}^n$, the unique solution $\vec{x}$ of $A\vec{x} = \vec{b}$ has entries given by

$$x_i = \frac{\det A_i(\vec{b})}{\det A}, i = 1, 2...n,$$

where $A_i(\vec{b})$ (or $A(i)$) is obtained from the matrix A by replacing the i-th column by $\vec{b}$.

Remark. Cramers Rule is not very efficient for large matrices.

Example 43 *Use Cramer's rule to solve the following linear system:*

$$\begin{aligned} x_1 - 2x_2 - 2x_3 &= 1 \\ x_2 + 2x_3 &= 0 \\ x_1 + x_3 &= 0 \end{aligned}$$

Solution. Let $A = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then

$$detA = a_{11}c_{11} + a_{21}c_{21} + a_{31}c_{31} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ 1 & 2 \end{bmatrix} = -1.$$

Note that

$$A_1(\vec{b}) = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, A_2(\vec{b}) = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}, A_3(\vec{b}) = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Hence

$$det(A_1(\vec{b})) = 1, det(A_2(\vec{b})) = 2, det(A_3(\vec{b})) = -1.$$

By Cramer's Rule,

$$x_1 = \frac{det(A_1(\vec{b}))}{detA} = -1, x_2 = \frac{det(A_2(\vec{b}))}{detA} = -2, x_3 = \frac{det(A_3(\vec{b}))}{detA} = 1.$$

Inverse Formula: If A is $n \times n$, then

$$A^{-1} = \frac{1}{\det A} adjA, \quad adjA = \begin{bmatrix} c_{11} & c_{21} & \dots & c_{n1} \\ c_{12} & c_{22} & \dots & c_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{1n} & c_{2n} & \dots & c_{nn} \end{bmatrix}.$$

5.1 Linear dependence and independence

Definition 13 *A set of vectors* $\{\vec{v}_1, \cdots, \vec{v}_m\}$ *in* $\mathbb{R}^n$ *is linearly independent if the vector equation*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{0}$$

has only the trivial solution. The set is said to be linearly dependent if there is a non-trivial solution to the vector equation.

Remark. $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m$ is called a linear combination of $\vec{v}_1, \cdots, \vec{v}_m$ with scalars $x_1, \cdots, x_m$. Thus $\{\vec{v}_1, \cdots, \vec{v}_m\}$ is linearly dependent if and only if one vector is a linear combination of other vectors.

Theorem 13 *The columns of* A *are linearly independent if and only if* $A\vec{x} = \vec{0}$ *ONLY has the trivial solution.*

Example 44 $\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$ *is linearly independent,* $\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\}$ *is linearly dependent.*

Elementary vectors (or standard basis vectors, or basic vectors): Elementary vector is a vector that has one component equals to 1 and all other components are 0. If the ith component is equal to 1, then we denote it by $\vec{e}_i$. A column of a matrix whose entries form an elementary vector is called an **elementary column**.

Theorem 14

1. *A set of two vectors is linearly dependent if and only if one of the vectors is a multiple of the other.*

2. *A set of two or more vectors is linearly dependent if and only if at least one vector may be written as a linearly combination of the others.*

3. *If a set contains more vectors than entries in each vector, then the set is linearly dependent.*

4. *If the zero vector is in a set of vectors, then the set of vectors is linearly dependent.*

5. *A set of vectors* $\{\vec{v}_1, \cdots, \vec{v}_m\}$ *in* $\mathbb{R}^n$ *is linearly independent if and only if the row-reduced echelon form of the matrix* $[\vec{v}_1 \ \cdots \ \vec{v}_m]$ *contains only distinct elementary columns.*

6. *A set of vectors* $\{\vec{v}_1, \cdots, \vec{v}_m\}$ *in* $\mathbb{R}^m$ *is linearly independent if and only if* $\det[\vec{v}_1 \cdots \vec{v}_m] \neq 0$.

Example 45 *Show that* $\{\vec{v_1} = (1,2,3), \vec{v_2} = (4,5,12), \vec{v_3} = (6,9,18)\}$ *is linearly dependent, and express one vector as a linear combination of other vectors.*

Solution:

$$[\vec{v}_1\, \vec{v}_2\, \vec{v}_3] = \begin{bmatrix} 1 & 4 & 6 \\ 2 & 5 & 9 \\ 3 & 12 & 18 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$.

5.2 Subspaces of $\mathbb{R}^m$

Definition 14 *A subspace is any subset* V *of* $\mathbb{R}^m$ *that satisfies:*

1. *For each* $\vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} \in V$ *(Closed under addition).*
2. *For each* $\vec{x} \in V$ *and scalar* c, $c\vec{x} \in V$ *(Closed under scalar multiplication.)*

Span of a set of vectors:

$$Span\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m; \quad \text{where } c_1, c_2, ..., c_m \text{ are scalars}\}.$$

Let S be a subspace, J be a subset of S. If span $J = S$, then J is called a spanning set of S, or S is generated by J.

Example 46

1. $S = \{\vec{0}\}$ *is a subspace.*
2. *Any span of a set of vectors is a subspace.*
3. *The null space of a matrix* A *is the set of all solutions to the homogeneous equation* $A\vec{x} = \vec{0}$ *and is a subspace of* $\mathbb{R}^n$*. The null space of* A *is denoted by* $NulA$*. It is also called* **solution space**.

$$NulA = \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}.$$

Example 47 1. $S = \{(a, a^2)|a \in \mathbb{R}\}$ *is not a subspace.*

2. $S = \{(a, 2a, 1)|a \in \mathbb{R}\}$ *is not a subspace.*

3. $S = \{(a, b)|a^2 = b^2, a, b \in \mathbb{R}\}$ *is not a subspace.*

4. $S = \{(a, b, c, d)|c = 2a - 3b, d = a + ba, b \in \mathbb{R}\}$ *is a subspace.*

5.3 Basis and dimension

Definition 15 *A basis for a subspace S is a linearly independent set of vectors that spans S. The number of vectors in a basis for a subspace S is called the dimension of S and is denoted by* $\dim S$.

Properties of basis:

- There is more than one basis for a subspace.
- A basis is the largest spanning set of linearly independent vectors for a subspace.

Example 48 *Let* $A = [\vec{a}_1\, \vec{a}_2\, \vec{a}_3\, \vec{a}_4\, \vec{a}_5] = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. *Let S be the subspace spanned (generated) by $J = \{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5\}$. Find a basis of S.*

Solution:

$$A = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 & -17 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the subset $\{\vec{a}_1, \vec{a}_4\}$ of J is a basis.

Example 49 *The set of elementary vectors in $\mathbb{R}^m$, $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_m\}$, is a basis of $\mathbb{R}^m$, and is called the standard basis of $\mathbb{R}^m$.*

Example 50 *Find a basis of* $S = \{(a, b, c, d)|c = 2a - 3b, d = a + ba, b \in \mathbb{R}\}$.

Example 51 *Let* $J = \{\vec{v}_1\, \vec{v}_2\}$ *be a set of linearly independent vectors in* $\mathbb{R}^4$ *with* $\vec{v}_1 = (1, 0, -1, 1)$, $\vec{v}_2 = (1, 1, 1, -1)$. *Extend* J *to form a basis of* $\mathbb{R}^4$.

Solution:

$$[\vec{v}_1\, \vec{v}_2\, \vec{e}_1\, \vec{e}_2\, \vec{e}_3\, \vec{e}_4] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

By choosing elementary columns we get $\{\vec{v}_1, \vec{v}_2, \vec{e}_1, \vec{e}_3\}$, which is a basis.

Theorem 15 *Let* A *be* $m \times n$. *If* $rank(A) = r$, *then the dimension of the solution space of* $A\vec{x} = \vec{0}$ *is* $n - r$.

Example 52 *Let*

$$A = [\vec{a}_1\, \vec{a}_2\, \vec{a}_3\, \vec{a}_4\, \vec{a}_5] = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find a basis for $NulA$, *the solution space of* $A\vec{x} = \vec{0}$.

Solution: Consider $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \vec{0}.$

$$\textit{Augmented matrix} = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 & -17 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 - 2x_3 + 17x_5 \\ x_2 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 17 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

Therefore, a basis is:

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 17 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

Theorem 16 *Let S be a subspace of $\mathbb{R}^m$ with $\dim S = n$, let $J \subset S$.*

1. *If J has more than n vectors, then J is dependent.*
2. *If J has less than n vectors, then span$(J) \neq S$.*
3. *If J has n vectors, then J is linearly independent iff span$(J) = S$.*
4. *If J has m vectors, then J is basis for $\mathbb{R}^m$ iff $\det(A) \neq 0$, where the columns of A consist of all vectors in J.*

Example 53 *Verify that the set of vectors $S = \{[1\ 2\ 3]^T, [-1\ 0\ 1]^T, [2\ 1\ \ -1]^T\}$ spans $\mathbb{R}^3$.*

5.4 Rank of a matrix

Definition 16 *Let $A = [\vec{a}_1 \ldots \vec{a_n}]_{m\times n}$.*

- *The columns of A span (or generate) a subspace of $\mathbb{R}^m$ called the column space of A and is denoted by ColA:*

$$ColA = Span\{\vec{a}_1, \ldots, \vec{a_n}\}.$$

  Its dimension is called column rank of A.

- *The rows of A span (or generate) a subspace of $\mathbb{R}^n$ called the row space of A. Its dimension is called row rank of A.*

Theorem 17 *Rank(A) = the column rank of A = the row rank of A.*

Example 54 *Let* $A = [\vec{a}_1\, \vec{a}_2\, \vec{a}_3\, \vec{a}_4\, \vec{a}_5] = \begin{bmatrix} 1 & -3 & 0 & 5 & 3 \\ 0 & 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Elementary columns are $\vec{a}_1\, \vec{a}_3$. *So* $\{\vec{a}_1, \vec{a}_3\}$ *is a basis of the column space of* A.

The Rank Theorem: If a matrix A has n columns then

$$dimColA + dimNulA = n$$

or

$$rankA + \dim NulA = n.$$

7.1 Definition of a Linear Transformation

Definition 17 *A transformation, (function or mapping),* T *from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ *is a rule that assigns a vector* $T(\vec{x}) \in \mathbb{R}^m$ *for each vector* $\vec{x} \in \mathbb{R}^n$.

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$\mathbb{R}^n$ *is called the domain and* $\mathbb{R}^m$ *the codomain of* T. *The vector* $T(\vec{x})$ *is called the image of* $\vec{x}$. *The set of all images of* T *is called the range of* T.

Example 55 *The matrix* A *is a transformation. We denote this by* T_A. *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}$.

(i) Find $T_A(\vec{b})$, where $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: Then $T_A(\vec{b}) = A\vec{b} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

(ii) Find $\vec{x}$ such that $T_A(\vec{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(iii) Is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in the range of $T_A(\vec{x})$?

Definition 18 *A transformation T is linear if*

1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T.

2. $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ and scalars c.

Properties: If T is linear, then

- $T(\vec{0}) = \vec{0}$.
- $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T and all scalars c, d.

Example 56 *T_A is linear for any matrix A.*

Example 57 *$T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\vec{x}) = \vec{x}$. Then T is linear and is called* **identity transformation** *I.*

Example 58 *$T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\vec{x}) = r\vec{x}$, r a scalar. Then T is linear.*

Example 59 *$T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x} + \vec{b}$, $\vec{b} \neq \vec{0}$. Then T is non-linear, since $T(\vec{0}) \neq \vec{0}$.*

Definition 19 *Two linear transformations S and T from $\mathbb{R}^n$ to $\mathbb{R}^m$ are equal if $T(\vec{x}) = S(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$.*

Theorem 18 *Let S and T be two linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. If for a basis $B = \{\vec{b_1}, ..., \vec{b_n}\}$, $S(\vec{b_i}) = T(\vec{b_i})$, $i = 1, ..., n$, then $S = T$.*

7.2 Algebra of a linear transformation

Theorem 19 *Let S and T be two linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, c be a scalar. Then $S \pm T$, $-S$, cS are linear, where*

$$(S \pm T)(\vec{x}) = S(\vec{x}) \pm T(\vec{x}); (-S)(\vec{x}) = -S(\vec{x}); (cS)(\vec{x}) = cS(\vec{x}).$$

Theorem 20 *Let S be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^k$. Then $T \circ S = T(S)$, the composition of T and S, is linear.*

Theorem 21 *Let L be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then the powers $L^0 = I$, $L^{k+1} = L(L^k)$ are linear, $k = 0, 1, 2, ...$*

Matrix of a linear transformation:

Theorem 22 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\vec{x}) = A\vec{x}$, for all $\vec{x} \in \mathbb{R}^n$. A is an $m \times n$ matrix:*

$$A = [T(\vec{e}_1)...T(\vec{e}_n)].$$

A is called the **matrix associated with** *T (or the standard matrix for the linear transformation T).*

Example 60 *Let* $T(\vec{x}) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix}$ *or* $T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$.

1) Find the matrix associated with T.

2) Solve $T(\vec{x}) = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$, *or find* $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *such that* $T(x_1, x_2) = (0, 1, 4)$.

1) $T(\vec{x}) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Thus $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}$.

2)

$$Augmentedmatrix = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 1 \\ 3 & -2 & 4 \end{bmatrix} \xrightarrow{R_2 + R_1, R_3 - 3R_1} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & 4 \end{bmatrix}$$

$$\xrightarrow{R_1 + 2R_2, R_3 - 4R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Theorem 23 *Let S and T be linear transformations with associated matrices A and B respectively. Then*

- $(S \pm T)(\vec{x}) = (A \pm B)(\vec{x})$;
- $(cS)(\vec{x}) = cA(\vec{x})$;
- $(-S)(\vec{x}) = -A(\vec{x})$;
- $(T \circ S)(\vec{x}) = BA(\vec{x})$.

Example 61 *Let $T(x, y, z) = (x + 2y - z, 3x - z, y + z)$. Find $2T^2 - T + 3I$.*

Solution: $(2T^2 - T + 3I)(\vec{x}) = (2A^2 - A + 3I)(\vec{x}) = 16x - 7z, 3x + 13y - 7z, 6x + y + 2z)$.

7.3 Kernel and image

Definition 20 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then the kernel (or nullspace of T) and image (or range) of T are defined as:*

$$ker(T) = \{\vec{x} | T(\vec{x}) = \vec{0}\}, \quad Im(T) = \{T(\vec{x}) | \vec{x}) \in \mathbb{R}^n\}.$$

$nullity(T) = \dim ker(T)$, $rank(T) = \dim Im(T)$.

Theorem 24 *$Ker(T)$ is a subspace of $\mathbb{R}^n$, $Im(T)$ is a subspace of $\mathbb{R}^m$.*

Example 62 *Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$. Find a basis for $Ker(T)$, and a basis of $Im(T)$.*

Definition 21 *A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be ONTO if each $\vec{b}$ in $\mathbb{R}^m$ is an image of at least one $\vec{x}$ in $\mathbb{R}^n$. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be One-to-one, (or 1:1), if each $\vec{b}$ in $\mathbb{R}^m$ is an image of at most one $\vec{x}$ in $\mathbb{R}^n$.*

The following results give ways of determining when a linear transformation is onto and one-to-one.

Theorem 25 *Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Then,*

1. *T_A is onto if and only if the columns of A span $\mathbb{R}^m$.*
2. *T_A is 1:1 if and only if the columns of A are linearly independent.*
3. *T_A is 1:1 if and only if $A\vec{x} = T_A(\vec{x}) = 0$ has only the trivial solution, i.e., $Ker(T) = \{\vec{0}\}$.*

Example 63 *Let* $A = \begin{bmatrix} 1 & 0 & 9 \\ 0 & 3 & 7 \end{bmatrix}$. *Is $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ ONTO, 1:1?*

Sol: T_A is ONTO, since the columns of A span $\mathbb{R}^2$. T_A is not 1:1, since the columns of A are linearly dependent.

Example 64 *Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by* $T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2) = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. *Is T ONTO, 1:1?*

Sol: T is not ONTO, since A has at most two pivots, the columns of A can not span $\mathbb{R}^3$. T_A is 1:1, since the columns of A are linearly independent.

Theorem 26 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear and A is the matrix associated with T. Then* $Im(T) = Col(A)$, $\dim(Im(T)) = rank(A)$.

Rank Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

$$\dim Ker(T) + rank(T) = n.$$

Proof. Let $\{v_1, \cdots, v_k\}$ be a basis for $ker(T)$. Then we can extend it to a basis of $\mathbb{R}^n$: $\{v_1, \cdots, v_k, v_{k+1}, \cdots v_n\}$. We only need to prove that $\{T(v_{k+1}), \cdots T(v_n)\}$ is a basis for $range(T)$.

Inverse: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with associated matrix A. If A is invertible, then the inverse of T is given by: $T{-}1(\vec{x}) = A^{-1}(\vec{x})$.

Theorem 27 *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with associated matrix A. Then the following are equivalent:*

1. *T is one-to-one,*

2. *T is onto,*

3. *T is invertible,*

4. $\det(A) \neq 0$.

Proof. " $\Rightarrow$ ": Let T be 1:1, then $ker(T) = \{0\}$. Let $B = \{v_1, ..., v_n\}$. Then $\{T(v_1), ..., T(v_n)\}$ is a basis of range(T). By Rank Theorem, $rank(T) = dim(\mathbb{R}^n)$. Thus $\{T(v_1), ..., T(v_n)\}$ is a basis of $\mathbb{R}^n$.

" $\Leftarrow$ " $rank(T) = n$, thus $nullity(T) = 0$. Thus 1:1.

4.4 An introduction to Eigenvalues

Definition 22 *An eigenvector of an $n \times n$ matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. $\vec{x}$ is called the eigenvector corresponding to λ.*

To determine whether a given value λ is an eigenvalue of a matrix A we need to find a non-zero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. This is the same as determining whether the matrix equation

$$(A - \lambda I)\vec{x} = 0$$

has a non-trivial solution.

Example 65 *Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$,$\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note that*

$$A\vec{u} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7\vec{u}, \quad A\vec{v} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\vec{v}, \quad A\vec{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \neq \lambda\vec{w}.$$

Thus $\vec{u}$ is an eigenvector corresponding to $\lambda = 7$, $\vec{v}$ is an eigenvector corresponding to $\lambda = -4$, $\vec{w}$ is not an eigenvector.

Example 66 *The eigenvalues of a triangular matrix are the entries on its main diagonal.*

Characteristic equation: $\det(A-\lambda I)$ is called the characteristic polynomial of A and

$$\det(A-\lambda I)=0$$

is called the characteristic equation.

Theorem 28 *The solutions of the characteristic equation are the eigenvalues of A.*

Example 67 *Let* $A=\begin{bmatrix} 4 & 5 \\ -1 & 0 \end{bmatrix}$. *Find all eigenvalues.*

Sol: $\det(A-\lambda I)=\lambda^2-4\lambda+5 \Rightarrow \lambda=2\pm i$.

Example 68 *Let* $A=\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. *Find all eigenvalues and eigenvectors.*

Solution: The characteristic polynomial is

$$\det(A-\lambda I)=\begin{vmatrix} 3-\lambda & 2 & -1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix}=(3-\lambda)^2(1-\lambda).$$

The solutions of the characteristic equation $\det(A-\lambda I)=0$ are 3,3,1.

When $\lambda=3$,

$$A-\lambda I=A-3I=\begin{bmatrix} 0 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}\xrightarrow{R_2+R_1}\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\xrightarrow{R_1+R_2}\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A-3I)\vec{x}=0$ has the solution

$$\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}=t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

When $\lambda=1$,

$$A-\lambda I=A-I=\begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}\xrightarrow{R_3-R_2}\begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\xrightarrow{R_1-\frac{1}{2}R_2}\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

8.1 Definitions and examples

Definition 23 *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear. A scalar λ is called an eigenvalue of T if there exists a nonzero vector $\vec{x}$ such that*

$$T(\vec{x}) = \lambda \vec{x}.$$

The vector $\vec{x}$ is called an eigenvector corresponding to the eigenvalue λ. We denote

$$E_\lambda = \{\vec{x} | T(\vec{x}) = \lambda \vec{x}\} = \{\vec{x} | (A - \lambda I)\vec{x} = \vec{0}\},$$

and call it the eigenspace of λ, where A is the matrix associated with T.

Theorem 29 *A real number λ is an eigenvalue of T if and only if λ is an eigenvalue of A.*

Example 69 *Let $T(x, y) = (0.95x + 0.03y, 0.05x + 0.97y)$. Find all eigenvalues and a basis to each eigenspace.*

Solution: Let $A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$.

a) Find all eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 0.95 - \lambda & 0.03 \\ 0.05 & 0.97 - \lambda \end{vmatrix} = \lambda^2 - 1.92\lambda + 0.92 = 0, \to$$

$\lambda = 1, 0.92$

b) Find a basis to each eigenspace:

When $\lambda = 1$,

$$A - \lambda I = A - I = \begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} -0.05 & 0.03 \\ 0 & 0 \end{bmatrix}.$$

Thus $(A - I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

The eigenspace has a basis $\left\{ \vec{V_1} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\} = \{(3,5)\}$.

When $\lambda = 0.92$,

$$A - \lambda I = A - 0.92I = \begin{bmatrix} 0.03 & 0.03 \\ 0.05 & 0.05 \end{bmatrix} \xrightarrow{R_2 - \frac{5}{3}R_1} \begin{bmatrix} 0.03 & 0.03 \\ 0 & 0 \end{bmatrix}.$$

Thus $(A - 0.93I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The eigenspace has a basis $\left\{ \vec{V_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \{(-1,1)\}$.

8.2 Diagonalization

Similar matrices: Two matrices A and B are similar if there is an invertible matrix P such that,

$$A = PBP^{-1}.$$

Theorem 30 *If $n \times n$ matrices A and B are similar, then they have the same characteristics polynomial and hence the same eigenvalues (with the same multiplicities).*

Proof.

$$\begin{aligned} \det(A - \lambda I) &= \det(PBP^{-1} - \lambda PP^{-1}) \\ &= \det[P(B - \lambda I)P^{-1}) \\ &= \det(P)\det(B - \lambda I)\det(P^{-1}) \\ &= \det(P)\det(B - \lambda I)\frac{1}{\det(P)} \\ &= \det(B - \lambda I). \end{aligned}$$

A diagonal matrix is a matrix with non zero values along its diagonal and zeros on its off diagonal entries. Well only be considering diagonal matrices that are square.

Definition 24 *If A is a square $n \times n$ matrix and A is similar to a diagonal matrix D then A is said to be diagonalizable.*

Theorem 31 *Let λ be an eigenvalue of A with corresponding eigenvector x.*

(1) For any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector x.

(2) If A is invertible, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector x.

(3) If A is invertible, then for any integer n, λ^{-n} is an eigenvalue of A^{-n} with corresponding eigenvector x.

Theorem 32 *If $\vec{v_1}, ..., \vec{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, . . . ,$ λ_r of an $n \times n$ matrix A, then the set $\{\vec{v_1}, ..., \vec{v_r}\}$ is linearly independent.*

Proof. We use induction.

Assume that when $r = k$, the set is linearly independent. When $r = k+1$, if on the contrary that the set is dependent, then

$$v_{k+1} = c_1 v_1 + \cdots c_k v_k.$$

Apply A to both sides,

$$\lambda_{k+1} v_{k+1} = c_1 \lambda_1 v_1 + \cdots c_k \lambda_k v_k.$$

We also have

$$\lambda_{k+1} v_{k+1} = c_1 \lambda_{k+1} v_1 + \cdots c_k \lambda_{k+1} v_k.$$

By subtraction,

$$0 = c_1(\lambda_1 - \lambda_{k+1}) v_1 + \cdots c_k(\lambda_k - \lambda_{k+1}) v_k.$$

Thus $c_1 = \cdots = c_k = 0$, i.e., $v_{k+1} = 0$, a contradiction.

Theorem 33 *(Diagonalization Theorem) Let A be an $n \times n$ matrix.*

- *A is diagonalizable if and only if A has n linearly independent eigenvectors. If $A = PDP^{-1}$, where D is a diagonal matrix, then the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.*
- *If A has n distinct eigenvalues, then A is diagonalizable.*
- *A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n, if and only if the dimension of the eigenspace for each eigenvalue equals the algebraic multiplicity of the eigenvalue. (Generally, the dimension of the eigenspace for each eigenvalue is less than or equal to the algebraic multiplicity of the eigenvalue).*

For an $n \times n$ matrix A, if A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to the eigenvalue λ_k, $k = 1, ..., p$, then the total collection of vectors in the sets $\mathcal{B}_1$,..., $\mathcal{B}_p$ forms an eigenvector basis of $\mathbb{R}^n$.

Example 70 $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ *is diagonalizable: 4 distinct eigenvalues.*

$B = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ *is not diagonalizable: $\lambda = 4$, one eigenvector* $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Example 71 *Let* $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

1) Find P and D such that $A = PDP^{-1}$.

2) Calculate A^4.

Sol: 1) $\det(A - \lambda I) = (\lambda - 5)(\lambda + 2)$.

When $\lambda = 5$: $\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

When $\lambda = -2$: $\vec{x} = \frac{x_2}{4} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Thus

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}; \text{ or } P = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}, D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}.$$

2) Let $P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$, then $P^{-1} = \frac{1}{7} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$.

$$A^4 = \{PDP^{-1}\}^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}^4 \frac{1}{7} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 625 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 364 & 261 \\ 348 & 277 \end{bmatrix}.$$

Diagonalization of symmetric matrix: A matrix A is symmetric if $A^T = A$.

Theorem 34 *Let A be symmetric.*

(i) If A is real, then all eigenvalues are real.

(ii) Any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. Using $x \cdot y = x^T y$.

Definition 25 *A square matrix Q is orthogonal if $Q^{-1} = Q^T$. A square matrix A is orthogonally diagonalizable if there exist an orthogonal matrix Q and a diagonal matrix D such that*

$$Q^T AQ = D.$$

Conditions under which a matrix is orthogonally diagonalizable:

Spectral Theorem (Principle Axes Theorem). Let A be a real square matrix. Then A is orthogonally diagonalizable if and only if A is symmetric.

Method to orthogonally diagonalize a symmetric matrix: Columns of Q consist of orthonormal bases of all eigenspaces.

Example 72 *Orthogonally diagonalize the matrix* $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Solution: Step 1: Find all eigenvalues: The characteristic polynomial is $-\lambda^3+6\lambda^2-9\lambda+4$. So $\lambda = 1, 4$.

Step 2: Find bases to each eigenspace:

Basis for E_1: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Basis for E_4: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Step 3: Find orthogonal bases to each eigenspace using Gram-Schmidt Process:

For E_1: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix} \right\}$.

Basis for E_4: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Step 4: Find orthonormal bases to each eigenspace:

For E_1: $\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right\}$.

Basis for E_4: $\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$.

Step 5: Construct Q and D:

$$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

8.4 Complex Eigenvalues and eigenvectors

Review: Complex Numbers

Complex number is

$$z = a + bi, \quad a \in \mathbb{R}, b \in \mathbb{R}, i^2 = -1.$$

Addition and multiplication:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

The conjugate of $z = a + bi$ is: $\bar{z} = a - bi$. We have

$$z\bar{z} = a^2 + b^2.$$

The modulus of z is defined by

$$|z| = \sqrt{a^2 + b^2}.$$

Properties

1. $\bar{z} = z$ if and only if z is real,
2. $\overline{z + w} = \bar{z} + \bar{w}$,
3. $\overline{zw} = \bar{z}\bar{w}$,
4. $|wz| = |w||z|$,
5. $|z + w| \leq |z| + |w|$ (triangle inequality),
6. let $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$ (De Moivre's Theorem).

Example 73 *Simplify* $\frac{2+3i}{4-5i}$.

Solution:

$$\frac{2 + 3i}{4 - 5i} = \frac{(2 + 3i)(4 + 5i)}{(4 - 5i)(4 + 5i)} = \frac{8 + 10i + 12i - 15}{4^2 + 5^2} = \frac{-7}{41} + i\frac{22}{41}.$$

Example 74 *Let* $z = \sqrt{3} + i$. *Calculate* z^6.

Sol: We write z as:

$$z = 2(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2e^{i\pi/6}.$$

$$z^6 = 2^6 e^{(i\pi/6)6} = 2^6 e^{i\pi} = 64(\cos\pi + i\sin\pi) = -64.$$

Complex vector:

A complex vector $\vec{z}$ in $\mathbb{C}^n$ is a vector with n entries of complex numbers.

$$\vec{x} = \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_n + ib_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \vec{a} + i\vec{b} = Re(\vec{z}) + iIm(\vec{z}).$$

The conjugate of $\vec{z}$ is

$$\bar{\vec{x}} = \begin{bmatrix} a_1 - ib_1 \\ a_2 - ib_2 \\ \vdots \\ a_n - ib_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} - i \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = Re(\vec{z}) - iIm(\vec{z}).$$

Dot product: Let $\vec{w} = (w_1, ..., w_m)$, $\vec{z} = (z_1, ..., z_m) \in \mathbb{C}^m$. We define

$$\vec{w} \cdot \vec{z} = w_1\bar{z}_1 + \cdots + w_m\bar{z}_m.$$

$$||\vec{w}|| = \sqrt{w_1\bar{w}_1 + \cdots + w_m\bar{w}_m} = \sqrt{|w_1|^2 + \cdots + |w_m|^2}.$$

Properties:

1. $\vec{w} \cdot \vec{w} \geq 0$, and $\vec{w} \cdot \vec{w} = 0$ if and only if $\vec{w} = \vec{0}$,

2. $\vec{w} \cdot \vec{z} = \overline{\vec{z} \cdot \vec{w}}$,

3. $(c\vec{w}) \cdot \vec{z} = c\vec{w} \cdot \vec{z}$, $\vec{w} \cdot (c\vec{z}) = \bar{c}\vec{w} \cdot \vec{z}$, where c is a complex number,

4. $\vec{w} \cdot \vec{w} = ||\vec{w}||^2$.

Complex matrix:

A complex matrix is a matrix with entries of complex numbers.

Properties of conjugates for vectors and matrices: Let A,B be matrix, $\vec{x}$ a vector and r a scalar.

1. $\overline{r\vec{x}} = \bar{r}\bar{\vec{x}}$.

2. $\overline{A\vec{x}} = \bar{A}\bar{\vec{x}}$.

3. $\overline{AB} = \bar{A}\bar{B}$.

4. $\overline{rA} = \bar{r}\bar{A}$.

Theorem 35 *If A is a real matrix, its eigenvalues come in conjugate pairs, if they are complex.*

Proof. Let λ be an eigenvalue of A, and $\vec{x}$ be an corresponding eigenvector, then

$$A\vec{x} = \lambda\vec{x}, \Rightarrow \bar{A}\bar{\vec{x}} = \bar{\lambda}\bar{\vec{x}}, A\bar{\vec{x}} = \bar{\lambda}\bar{\vec{x}}.$$

Thus $\bar{\lambda}$ is an eigenvalue of A, and $\bar{\vec{x}}$ is an corresponding eigenvector.

Theorem 36 *Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi (b \neq 0)$ and associated eigenvector $\vec{x}$ in $\mathbb{C}^2$. Then,*

$$A = PCP^{-1}, \quad P = [Re(\vec{x}) \ Im(\vec{x})], \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Example 75 *Let* $A = \begin{bmatrix} 4 & 5 \\ -1 & 0 \end{bmatrix}$.

1) Find all eigenvalues.

2) Find a basis of each eigenspace.

3) Find P and C such that $A = PCP^{-1}$.

Sol: 1) $\det(A - \lambda I) = \lambda^2 - 4\lambda + 5 \Rightarrow \lambda = 2 \pm i$.

2) When $\lambda = 2+i$: $(A-\lambda I)\vec{x} = \vec{0} \Rightarrow \vec{x} = x_2 \begin{bmatrix} -2-i \\ 1 \end{bmatrix}$.

3) Let $P = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

Example 76 *Let* $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & -2 & 3 \end{bmatrix}$.

1) Find all eigenvalues in $\mathbb{C}$.

2) Find a basis to each eigenspace.

Solution: The characteristic polynomial is

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 5 \\ 0 & -2 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1-\lambda & 5 \\ -2 & 3-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 4\lambda + 13).$$

Let $(2-\lambda)(\lambda^2 - 4\lambda + 13) = 0$, we imply that

$$\lambda = 2, 2+3i, 2-3i.$$

2) When $\lambda = 2$,

$$A - \lambda I = A - 2I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 5 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_2 + R_1, R_3 + 2R_1} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\xrightarrow{R_1 - \frac{1}{2}R_2, R_3 - \frac{1}{2}R_2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A-2I)\vec{x} = 0$ has the solution $x_2 = x_3 = 0$, x_1=free.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenspace has a basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

When $\lambda = 2 - 3i$,

$$A - \lambda I = A - (2-3i)I = \begin{bmatrix} 3i & 1 & 1 \\ 0 & -1+3i & 5 \\ 0 & -2 & 1+3i \end{bmatrix} \xrightarrow{5R_1, R_3 + \frac{2}{-1+3i}R_2} \begin{bmatrix} 15i & 5 & 5 \\ 0 & -1+3i & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{bmatrix} 15i & 6-3i & 0 \\ 0 & -1+3i & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - (2-3i)I)\vec{x} = 0$ has the solution $x_1 = \frac{6-3i}{15i}x_2 = \frac{-1-2i}{5}x_2$, $x_3 = \frac{-1+3i}{-5}x_2$, x_2=free.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{5}x_1 \begin{bmatrix} 1+2i \\ 1 \\ -1+3i \end{bmatrix}.$$

The eigenspace has a basis $\left\{ \begin{bmatrix} 1+2i \\ 1 \\ -1+3i \end{bmatrix} \right\}$.

By conjugation, the eigenspace to the eigenvalue $2+3i$ has a basis $\overline{\left\{ \begin{bmatrix} 1+2i \\ 1 \\ -1+3i \end{bmatrix} \right\}} =$ $\left\{ \begin{bmatrix} 1-2i \\ 1 \\ -1-3i \end{bmatrix} \right\}$.

Theorem 37 *(Diagonalization Theorem) Let A be an $n \times n$ matrix with complex entries.*

- *Nonzero eigenvectors that correspond to distinct eigenvalues are linearly independent.*
- *A is diagonalizable if and only if A has n linearly independent eigenvectors. If $P^{-1}AP = D$, where D is a diagonal matrix, then the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.*

- *If A has n distinct eigenvalues, then A is diagonalizable.*

Example 77 *Let* $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

1) Find all eigenvalues in $\mathbb{C}$.

2) Find a basis to each eigenspace.

3) Is A diagonalizable? If yes, find P and D; if no, explain why.

Solution: 1) The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = (i-\lambda)(\lambda^2+1).$$

We imply that $\lambda = i, i, -i$.

2) When $\lambda = i$,

$$A - iI = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 0 & -1 & -i \\ 0 & -i & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{R_2 - iR_1} \begin{bmatrix} 0 & -1 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - iI)\vec{z} = 0$ has the solution $-z_2 - iz_3 = 0$, z_1=free.

$$\vec{z} = (z_1, z_2, z_3) = z_1(1,0,0) + z_3(0,-i,1).$$

The eigenspace has a basis $\{(1,0,0),(0,-i,1)\}$.

When $\lambda = -i$,

$$A - \lambda I = A - (-i)I = \begin{bmatrix} 2i & 0 & 0 \\ 0 & i & 1 \\ 0 & -1 & i \end{bmatrix} \xrightarrow{R_3 - iR_2} \begin{bmatrix} 2i & 0 & 0 \\ 0 & i & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - (2-3i)I)\vec{z} = 0$ has the solution $z_1 = 0$, $iz_2 + z_3 = 0$. $\vec{z} = z_2(0,1,-i)$. The eigenspace has a basis $\{(0,1,-i)\}$.

Thus

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -i & 1 \\ 0 & 1 & -i \end{bmatrix}, D = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}.$$

Hermitian matrix: A square matrix A is Hermitian if $A = A^H$, where $A^H = \bar{A}^T$. A nonsingular matrix P is unitary if $p^{-1} = P^H$.

Theorem 38 *Let A be a Hermitian matrix. Then*

- *All eigenvalues are real.*
- *Eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Theorem 39 *A nonsingular matrix P is unitary if and only if its columns form an orthonormal set.*

Theorem 40 *There exists a unitary P such that $P^{-1}AP = D$ if and only if A is Hermitian.*

Example 78 *Let* $A = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

1) Show that A is Hermitian.

2) Find a a unitary P such that $P^{-1}AP = D$