Math. 2008–2009 B, Winter 2013 – Assignment #2
Solution

1. Find the (absolute) minimum and maximum values of $f(x, y) = e^y(x^2 + y^2)$ on the region $2x^2 + y^2 \leq 1$.

Solution. If the extreme values are achieved at any points inside the region, i.e. on $2x^2 + y^2 < 1$, then any such point should be a critical point of the function. So, we first need to locate such points inside the region. For that, we solve

$$f_x = 0 \qquad \textit{and} \qquad f_y = 0,$$

the first one gives

$$2xe^y = 0 \Rightarrow x = 0$$

and so the second equation reads

$$f_y = 0 \Rightarrow e^y(x^2 + y^2 + 2y) = e^y(y^2 + 2y) = 0 \Rightarrow y = 0 \;\; \textit{or} \;\; y = -2.$$

So, we obtain two solutions $(0, 0)$ and $(0, -2)$ for above equations, but since the latter does not lie in our region, we will not consider it.

Next, we look at the boundary of the region, i.e. the curve $2x^2 + y^2 = 1$, and try to find the extreme values of f among the points on this curve. For that, we use the method of Lagrange Multipliers; here f is the function to be optimized, and the constraint is $g(x, y) := 2x^2 + y^2 = 1$. So, we solve the system of equations

$$\nabla f = \lambda \nabla g \qquad \textit{and} \qquad g(x, y) = 2x^2 + y^2 = 1$$

which gives

$$f_x = \lambda\, g_x \Rightarrow 2xe^y = 4\lambda x$$

$$f_y = \lambda\, g_y \Rightarrow e^y(x^2 + y^2 + 2y) = 2\lambda y$$

$$2x^2 + y^2 = 1.$$

Now, the first equation gives $x = 0$ or $e^y = 2\lambda$. For $x = 0$ the third equation gives $y = \pm 1$. And if $e^y = 2\lambda$, the second equation will read $x^2 + y^2 + 2y = y$, that is

$x^2 = -y^2 - y$, and so the third equation gives $-y^2 - 2y = 1$, which has one solution $y = -1$, and $x = 0$. (We had found this point already!)

Finally, we evaluate, and compare the values of f at these candidate points:

$$f(0,0) = 0 \quad , \quad f(0,-1) = e^{-1} \quad , \quad f(0,1) = e\,.$$

Therefore, 0 and e are the minimum and maximum values of f on the given region.

2. At what point(s) on the surface $z^2 = x - 2y^2$, the tangent plane is parallel to the plane $-2x + 8y - 2z = 17$?

Solution. Two planes are parallel if their normal vectors are parallel. Since the gradient vector at any point on a surface is normal to the tangent plane to the surface at that point, we need to find the points on the surface $z^2 = x - 2y^2$ at which the gradient vector is parallel to the normal vector of the plane $-2x + 8y - 2z = 17$, which is $\langle -2, 8, -2 \rangle$.

The gradient vector of the above surface at a point (x, y, z) is $\langle 1, -4y, -2z \rangle$; for this vector to be parallel to $\langle -2, 8, -2 \rangle$, we must have $y = 1$, and $z = \frac{-1}{2}$. Hence, $(\frac{9}{4}, 1, \frac{-1}{2})$ is the only point on the surface at which the tangent plane is parallel to the plane $-2x + 8y - 2z = 17$.

3. Find and classify the relative extrema and/or saddle points of the function

$$f(x,y) = x^2 + 2y^2 + x^2y + 3.$$

Solution.

$$\frac{\partial f}{\partial x} = 2x + 2xy \qquad \frac{\partial f}{\partial y} = 4y + x^2$$

Now solving

$$2x + 2xy = 0 \qquad \textit{and} \qquad 4y + x^2 = 0$$

we obtain the points $(0,0)$, $(-2,-1)$, and $(2,-1)$. Moreover, we have

$$f_{xx} = 2 + 2y \qquad f_{yy} = 4 \qquad f_{xy} = 2x$$

and so $D(x,y) = 8 + 8y - 4x^2$.

At $(0,0)$ we have $D(0,0) = 8 > 0$ and $f_{xx}(0,0) = 2 > 0$. So, f has a local minimium at this point.

At $(-2,-1)$ we have $D(-2,-1) = -16 < 0$. So, f has a saddle point at this point.

At $(2,-1)$ we have $D(2,-1) = -16 < 0$. So, f has a saddle point at this point.

4. Suppose a sound with frequency f is emitted by an object moving along a straight line with speed u, and a listener is traveling along the same line in the opposite direction with speed v. Then the frequency F heard by the listener is given by

$$F = \left(\frac{c - v}{c + u}\right) f$$

where $c \approx 1100$ ft/sec (This is the **Doppler effect**). Suppose that a railroad train is traveling at 120 ft/sec and accelerating at the rate of 2 ft/sec^2 and that a note emitted by the locomotive whistle has the frequency 800 Hz. If a passenger is on a train that is moving at 80 ft/sec in the direction opposite to that of the first train and accelerating at the rate of 4 ft/sec^2, how fast the frequency of the note is changing for the passenger?

Solution. From the problem we have $u = 120$, $\frac{du}{dt} = 2$, $v = 80$, $\frac{dv}{dt} = 4$, and $f = 800$. And we are looking for $\frac{dF}{dt}$. By the chain rule we get

$$\begin{aligned} \frac{dF}{dt} &= \frac{\partial F}{\partial v}\frac{dv}{dt} + \frac{\partial F}{\partial u}\frac{du}{dt} = \frac{-800}{c+u}\frac{dv}{dt} + 800\frac{v-c}{(c+u)^2}\frac{du}{dt} \\ &= (\frac{-800}{1100+120})(4) + 800(\frac{80-1100}{(1100+120)^2})(2) \approx -3.72\,. \end{aligned}$$

The frequency heard by the passenger is droping at a rate of approximately 3.72 Hz/sec.

5. Marine biologists have determined that when a shark detects the presence of blood in the water, it will swim in the direction in which the concentration of the blood increases most rapidly. Suppose that in a certain case, the concentration of blood at a point $P(x, y)$ on the surface of seawater is given by

$$C(x, y) = 10^8 - 20x^2 - 40y^2 ,$$

where x and y are measured in meters in a rectangular coordinate system with the blood source at the origin.

Suppose a shark is at the point $(100, 500)$ on the surface of seawater when it first detects the presence of blood in the water. Find the equation of the shark's path towards the blood source.

(You can use the fact that if $y = f(x)$ satisfies $f'(x) = \frac{a}{x} f(x)$, then $f(x) = Cx^a$ for some constant C.)

Solution. First note that at each point the blood concentration increases most rapidly in the direction of $\nabla(C)$. Now, assume that the path which the shark is taking is given by the vector function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. Then since the tangent vector $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$ at each point on the path is parallel to the gradient vector $\nabla(C)(x, y) = \langle -40x, -80y \rangle$, we obtain

$$\frac{y'(t)}{x'(t)} = \frac{-80y}{-40x} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2y}{x},$$

which gives (from the hint at the end of question) that $y = Cx^2$ for some constant C. Since the point $(100, 500)$ is on the path, we find $C = 0.05$, and so $y = 0.05x^2$ is the equation of the shark's path towards the blood source.