MATH3705 Notes - By Eric Hua

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1 I LAPLACE TRANSFORM

1.1 Definition of LT

1.1.1 Definition

The Laplace Transform is widely used in engineering applications, such as solving linear ordinary differential equations. It transforms the equation in "t-space" to one in "s-space". This makes the problem much easier to solve.

Let $f(t)$ be a function defined on $[0, \infty)$. We may assume $f(t) = 0$ when $t < 0$. The Laplace transform (LT) of $f(t)$ is the function $F(s)$, defined by:

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

This is named for Pierre-Simon Laplace, one of the best French mathematicians in the mid-to-late 18th century.The LT transforms functions of t to functions of another variable s.

If $F(s)$ is the Laplace transform of $f(t)$, then $f(t)$ is the **inverse Laplace transform** of $F(s)$:

$$f(t) = L^{-1}\{F(s)\}.$$

Proposition 1. *The LT and L^{-1} are **linear**:*

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

$$L^{-1}\{aF(s) + bG(s)\} = aL^{-1}\{F(s)\} + bL^{-1}\{G(s)\}$$

Examples of Laplace Transforms	
$f(t)$ for $t \geq 0$	$\mathcal{L}(f)$
1	$\frac{1}{s}$
$e^{at}, \quad s > a$	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$

For example,

$$L\{1\} = \int_0^\infty e^{-st}\,dt = \left\{-\frac{1}{s}e^{-st}\right\}_0^\infty = \frac{1}{s}.$$

Since $\int e^{bt}\sin at dt = \frac{e^{bt}}{a^2+b^2}(b\sin at - a\cos at)$, we have

$$L\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt = \left\{\frac{e^{-st}}{a^2+s^2}(-s\sin at - a\cos at)\right\}_0^\infty = \frac{a}{s^2+a^2}.$$

$$L\left\{e^{at}\right\} = \int_0^\infty e^{(a-s)t}\,dt = \left\{-\frac{1}{s-a}e^{(a-s)t}\right\}_0^\infty = \frac{1}{s-a}.$$

1.1.2 Existence

For which functions f is the LT actually defined on? We want the indefinite integral to converge, of course.

A function $f(t)$ is piecewise continuous on a finite interval $[a, b]$ if $[a, b]$ can be subdivided into a finite number of subintervals such that f is continuous in each of the subintervals, and it approaches a finite limit when t approaches an end of any of the subintervals. Function f is piecewise continuous on an infinite interval if it is piecewise continuous on any finite subinterval of its domain.

A function $f(t)$ is of **exponential order** α if there exist constants t_0 and M such that

$$|f(t)| < Me^{\alpha t}, \quad \text{for all } t > t_0.$$

For example, t^n and e^t are of exponential order, e^{t^2} is not of exponential order.

Theorem 1. *Suppose that $f(t)$ is piecewise continuous and of exponential order with*

$$|f(t)| < Me^{\alpha t}, \quad \textit{for all } t > t_0.$$

Then the Laplace transform of $f(t)$ exists for all $s > \alpha$.

Proof.

$$|F(s)| \leq M \int_0^\infty e^{(\alpha - s)t}\, dt = \frac{M}{s-\alpha} < \infty.$$

Remark. The condition in this theorem is sufficient but not necessary. For example, the function $f(t) = t^n, -1 \leq n < 0$ is not piecewise continuous. But the Laplace transform of $f(t)$ exists.

Remark. If $f(t)$ satisfies the conditions in the theorem above, then $F(s) \to 0$ when $s \to \infty$.

Example 1. $L\left\{e^{t^2}\right\}$ *does not exist.*

Proof.

$$L\left\{e^{t^2}\right\} \geq e^{-s^2/4} \int_0^\infty e^{(t-s/2)^2}\, dt \geq e^{-s^2/4} \int_0^\infty 1\, dt = \infty.$$

1.1.3 Laplace Transform of Periodic Functions

A function $f(t)$ is periodic if there exists $\omega > 0$ such that $f(t+\omega) = f(t)$ for all t. ω is a period of f. The smallest positive period ω of f is called the fundamental period of f, and f is called ω-periodic.

Theorem 2. *If f is called ω-periodic for $t \geq 0$, then*

$$L\{f(t)\} = \frac{1}{1-e^{-\omega s}} \int_0^\omega e^{-st} f(t)\, dt.$$

Proof.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \sum_{n=0}^\infty \int_{n\omega}^{(n+1)\omega} e^{-st} f(t)\, dt = \sum_{n=0}^\infty \int_0^\omega e^{-s(x+n\omega)} f(x)\, dx,$$

where $x = t - n\omega$.

Example 2. *If* $f(t) = \begin{cases} t, & 0 \leq t < 1; \\ 0, & 1 \leq t < 2. \end{cases}$ *and 2-periodic, then*

$$L\{f(t)\} = \frac{1-(1+s)e^{-s}}{s^2(1-e^{-2s})}.$$

Example 3. *Half-rectified wave:* $f(t) = \begin{cases} \sin t, & 2k\pi \le t < (2k+1)\pi; \\ 0, & (2k+1)\pi \le t < (2k+2)\pi. \end{cases}$ *Then*

$$L\{f(t)\} = \frac{1}{(s^2+1)(1-e^{-\pi s})}.$$

Example 4. *Full-rectified wave:* $f(t) = \begin{cases} \sin t, & 2k\pi \le t < (2k+1)\pi; \\ -\sin t, & (2k+1)\pi \le t < (2k+2)\pi. \end{cases}$ *Then*

$$L\{f(t)\} = \frac{1}{(s^2+1)} coth \frac{\pi s}{2}.$$

1.1.4 The Gamma Function

To find Laplace transform of t^n where n is not an integer, we need the Gamma function:

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx \quad (p > 0).$$

Properties:

- $\Gamma(1) = 1$.
- It is divergent when $p = 0$.
- $\Gamma(p+1) = p\Gamma(p)$ for any p by integration by parts, except for 0, -1, -2, -3, .
- $\Gamma(n+1) = n!$

Example 5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Example 6. $\Gamma(\frac{2k+1}{2}) = \frac{(2k-1)!!}{2^k}\sqrt{\pi}$.

$\Gamma(-\frac{2k-1}{2}) = \frac{2^k}{(2k-1)!!}\sqrt{\pi}$.

Theorem 3. *For any* $p > -1$, $L\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}$.

Proof. Let $x = st$.

Example 7. $L\{t^{0.5}\} = \frac{\Gamma(0.5+1)}{s^{0.5+1}} = \frac{1}{2s^{0.5+1}}\sqrt{\pi}$.

$L\{t^{-0.5}\} = \frac{\Gamma(-0.5+1)}{s^{-0.5+1}} = \sqrt{\frac{\pi}{s}}$.

1.2 DIFFERENTIATION, INTEGRATION AND LT

1.2.1 LT of the derivative of a function

In the definition of the LT, replace $f(t)$ by it's derivative $f'(t)$:

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt.$$

Now integrate by parts ($u = e^{-st}$, $dv = f'(t)\, dt$):

$$\int_0^\infty e^{-st} f'(t)\, dt = f(t)e^{-st}|_0^\infty - \int_0^\infty f(t) \cdot (-s) \cdot e^{-st}\, dt = -f(0) + sL\{f(t)\}.$$

Therefore, if $F(s)$ is the LT of $f(t)$ then $sF(s) - f(0)$ is the LT of $f'(t)$:

$$L\{f'(t)\} = sL\{f(t)\} - f(0). \tag{1}$$

Replace f by f' in (1),

$$L\{f''(t)\} = sL\{f'(t)\} - f'(0), \tag{2}$$

and apply (1) again:

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0), \tag{3}$$

This is called a **derivative theorem** for the LT.

By induction, we have

Theorem 4. *Suppose f and its derivatives of order up to $n-1$ are continuous and of exponential order with $|f^{(j)}(t)| \leq Ke^{at}$ for all $t > M$, $0 \leq j \leq n-1$, and that $f^{(n)}$ is piece-wise continuous, then $L\{f^{(n)}\}$ exists for $s > a$ and*

$$L\left\{f^{(n)}(t)\right\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

Using (1) and (3), the LT of any constant coefficient ODE

$$ax''(t) + bx'(t) + cx(t) = f(t)$$

is

$$a(s^2 L\{x(t)\} - sx(0) - x'(0)) + b(sL\{x(t)\} - x(0)) + cL\{x(t)\} = F(s),$$

where $F(s) = L\{f(t)\}$. In particular, the LT of the solution, $X(s) = L\{x(t)\}$, satisfies

$$X(s) = \frac{F(s) + asx(0) + ax'(0) + bx(0)}{as^2 + bs + c}.$$

Note that the denominator is the characteristic polynomial of the DE.

Example 8. *Using the Laplace transform solve the differential equation*

$$f'' - 4f' + 3f = 1$$

with boundary conditions $f(0) = f'(0) = 0$.

Solution: Take the Laplace transform of the equation. Since $f'(0) = f(0) = 0$, if $\mathcal{L}(f) = F(s)$ then $\mathcal{L}(f') = sF(s)$ and $\mathcal{L}(f'') = s^2F(s)$. Thus,

$$s^2F - 4sF + 3F = \frac{1}{s}$$

and so

$$\begin{aligned} (s^2 - 4s + 3)F &= \frac{1}{s} \\ F &= \frac{1}{s}\frac{1}{s^2 - 4s + 3} \end{aligned} \tag{4}$$

and, since $s^2 - 4s + 3 = (s-3)(s-1)$, this gives

$$F = \frac{1}{s(s-3)(s-1)}$$

Before we can invert this, we need to do a partial fraction expansion.

$$\begin{aligned} \frac{1}{s(s-3)(s-1)} &= \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-1} \\ 1 &= A(s-3)(s-1) + Bs(s-1) + Cs(s-3) \end{aligned} \tag{5}$$

So substituting in $s = 0$ we get $A = 1/3$, $s = 3$ gives $B = 1/6$ and $s = 1$ gives $C = -1/2$. Hence

$$F = \frac{1}{3s} + \frac{1}{6(s-3)} - \frac{1}{2(s-1)}$$

and so

$$f(t) = \frac{1}{3} + \frac{1}{6}e^{3t} - \frac{1}{2}e^{t}.$$

Example 9. *Using the Laplace transform solve the differential equation*

$$f'' - 4f' + 3f = 0$$

with boundary conditions $f(0) = 1$ *and* $f'(0) = 1$.

Solution: In this example there are non-zero boundary conditions. Since

$$\mathcal{L}(f') = sF - f(0) \tag{6}$$
$$\mathcal{L}(f'') = s^2F - sf(0) - f'(0) \tag{7}$$

the subsidiary equation in this case is

$$s^2F - s - 1 - 4sF + 4 + 3F = 0$$

so

$$(s^2 - 4s + 3)F = s - 3.$$

Hence

$$F = \frac{1}{s-1}$$

and

$$f(t) = e^t$$

1.2.2 LT of the integral of a function

Theorem 5. *If f is continuous and $L\{f\}$ exists, then*

$$L\left\{\int_0^t f(x)dx\right\} = \frac{L\{f\}}{s}.$$

Equivalently,

$$L^{-1}\left\{\frac{L\{f\}}{s}\right\} = \int_0^t f(x)dx.$$

Proof. Let $g(t) = \int_0^t f(x)dx$. Then

$$L\{f(t)\} = L\{g'(t)\} = sL\{g(t)\} - g(0) = sL\{g(t)\}.$$

Example 10.

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a},$$
$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{\sin ax}{a}dx = \frac{1-\cos at}{a^2},$$
$$L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \int_0^t \frac{1-\cos ax}{a^2}dx = \frac{at - \sin at}{a^3}.$$

Example 11. *Solve the integral equation $f(t) + \int_0^t f(x)dx = 1$.*

Solution: Taking the Laplace transform we get

$$F(s) + \frac{F(s)}{s} = \frac{1}{s}, \Rightarrow F(s) = \frac{1}{s+1}, \Rightarrow f(t) = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}.$$

1.2.3 The Derivative of LT of a Function

Differentiate the definition of the LT with respect to s:

$$F'(s) = -\int_0^\infty e^{-st} t f(t)\, dt.$$

Repeating this:

$$\frac{d^n}{ds^n} F(s) = (-1)^n \int_0^\infty e^{-st} t^n f(t)\, dt. \qquad (8)$$

This gives:

Theorem 6. *If $f(t)$ is piecewise continuous and of exponential order, and $L\{f(t)\} = F(s)$, then*

$$L\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

Equivalently,

$$L^{-1}\left\{F(s)\right\} = (-1)^n \frac{L^{-1}\left\{F^{(n)}(s)\right\}}{t^n}.$$

These formulas are used in the following cases:

(i) Find the Laplace transforms of function with the forms $t^n f(t)$, when the transform of $f(t)$ is easy to find.

(ii) Find the inverse transform of $F(s)$ if the inverse transform of the derivative of $F(s)$ is easy to find.

As we know the LT of $f(t) = e^{at}$ is $F(s) = (s-a)^{-1}$. By the theorem we have

Example 12.

$$L\left\{te^{at}\right\} = -F'(s) = (s-a)^{-2}, \quad L\left\{t^2 e^{at}\right\} = F''(s) = 2 \cdot (s-a)^{-3},$$
$$L\left\{t^3 e^{at}\right\} = -F'''(s) = 2 \cdot 3 \cdot (s-a)^{-4}, \quad \ldots ,$$

and in general

$$L\left\{t^n e^{at}\right\} = (-1)^n F^{(n)}(s) = n! \cdot (s-a)^{-n-1}. \qquad (9)$$

Example 13. *Using the Laplace transform solve the differential equation*

$$f'' - 4f' + 3f = 2e^t$$

with boundary conditions $f(0) = f'(0) = 0$.

Solution: This time we have $\mathcal{L}(2e^t) = 2/(s-1)$ on the right hand side. This means that the subsidiary equation is

$$(s^2 - 4s + 3)F = \frac{2}{s-1}$$

so

$$F = \frac{2}{(s-1)^2(s-3)}$$

We need to do partial fractions again, but this is one of those cases with a repeated root:

$$\frac{1}{(s-1)^2(s-3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-3}$$

and multiplying across

$$1 = A(s-1)(s-3) + B(s-3) + C(s-1)^2$$

so $s = 1$ gives $B = -1/2$ and $s = 3$ gives $C = 1/4$. No value of s gives A on its own, so wee try $s = 2$:

$$1 = -A + \frac{1}{2} + \frac{1}{4}$$

which means that $A = -1/4$. Hence

$$F = -\frac{1}{2(s-1)} - \frac{1}{(s-1)^2} + \frac{1}{2(s-3)}$$

and

$$f = -\frac{1}{2}e^t - te^t + \frac{1}{2}e^{3t}$$

Example 14. *Using the Laplace transform solve the differential equation*

$$y'' - 2ay' + a^2y = 0$$

with boundary conditions $y'(0) = 1$ and $y(0) = 0$. a is some real constant.

Solution: Taking the Laplace transform we get

$$s^2Y - 1 - 2aY + a^2Y = 0$$

and hence

$$Y = \frac{1}{(s-a)^2}$$

which means that

$$y = te^{at}$$

Example 15. *Using LT to solve the DE*

$$x' + x = t^{100}e^{-t}, \quad x(0) = 0.$$

Note this would be highly impractical to solve using undetermined coefficients. (You would have 101 *undetermined coefficients to solve for!)*

Solution: First, we compute the LT of the solution to the DE. The LT of the LHS: by (1),

$$L\{x' + x\} = sX(s) + X(s),$$

where $F(s) = L\{f(t)\}(s)$. For the LT of the RHS, let

$$F(s) = L\left\{e^{-t}\right\} = \frac{1}{s+1}.$$

By (8),

$$\frac{d^{100}}{ds^{100}}F(s) = L\left\{t^{100}e^{-t}\right\} = \frac{d^{100}}{ds^{100}}\frac{1}{s+1}.$$

The first several derivatives of $\frac{1}{s+1}$ are as follows:

$$\frac{d}{ds}\frac{1}{s+1} = -\frac{1}{(s+1)^2}, \quad \frac{d^2}{ds^2}\frac{1}{s+1} = 2\frac{1}{(s+1)^3}, \quad \frac{d^3}{ds^3}\frac{1}{s+1} = -62\frac{1}{(s+1)^4},$$

and so on. Therefore, the LT of the RHS is:

$$\frac{d^{100}}{ds^{100}}\frac{1}{s+1} = 100!\frac{1}{(s+1)^{101}}.$$

Consequently,

$$X(s) = 100!\frac{1}{(s+1)^{102}}.$$

Using (9), we can compute the ILT of this:

$$x(t) = L^{-1}\{X(s)\} = L^{-1}\left\{100!\frac{1}{(s+1)^{102}}\right\} = \frac{1}{101}L^{-1}\left\{101!\frac{1}{(s+1)^{102}}\right\} = \frac{1}{101}t^{101}e^{-t}.$$

1.2.4 The Integral of LT of a Function

Theorem 7. *Let $f(t)$ be piecewise continuous and of exponential order, and $L\{f(t)\} = F(s)$. If $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists, then*

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{+\infty} F(x)dx.$$

Equivalently,

$$L^{-1}\{F(s)\} = tL^{-1}\left\{\int_s^{+\infty} F(x)dx\right\}.$$

Proof. Let $g(t) = \frac{f(t)}{t}$. Then $f(t) = tg(t)$, $F(s) = L\{f\} = -G'(s)$.

These formulas are used in the following cases:

(i) Find the Laplace transforms of function with the forms $\frac{f(t)}{t}$, when the transform of $f(t)$ is easy to find.

(ii) Find the inverse transform of $F(s)$ if the inverse transform of the integral of $F(s)$ is easy to find.

Example 16.

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^{+\infty} \frac{a}{x^2+a^2}dx = \frac{\pi}{2} - \arctan(s/a), \quad a > 0.$$

Example 17.

$$\begin{aligned} L^{-1}\left\{\frac{1}{(s-1)^3}\right\} &= tL^{-1}\left\{\int_s^{+\infty} \frac{1}{(x-1)^3}dx\right\} \\ &= -t/2L^{-1}\left\{\frac{1}{(s-1)^2}\right\} \\ &= (-t/2)tL^{-1}\left\{\int_s^{+\infty} \frac{1}{(x-1)^2}dx\right\} \\ &= t^2/2L^{-1}\left\{\frac{1}{s-1}\right\} \\ &= \frac{t^2e^t}{2} \end{aligned}$$

1.3 SHIFTING THEOREMS

1.3.1 The First Shifting Theorem

Theorem 8. *If $F(s)$ exists for $s > c \geq 0$, then, for any constant $a < s - c$,*

$$L\left\{e^{at} f(t)\right\} = F(s-a).$$

Equivalently,

$$L^{-1}\left\{F(s-a)\right\} = e^{at} L^{-1}\left\{F(s)\right\}.$$

Example 18.

$$L\left\{e^{bt} \sin at\right\} = \frac{a}{(s-b)^2 + a^2}.$$

Example 19.

$$L^{-1}\left\{\frac{s}{(s+3)^2+1}\right\} = L^{-1}\left\{\frac{s+3-3}{(s+3)^2+1}\right\} = e^{-3t} L^{-1}\left\{\frac{s-3}{s^2+1}\right\} = e^{-3t}(\cos t - 3\sin t).$$

1.3.2 The Second Shifting Theorem

Define the unit step (Heaviside) function by

$$u(t-a) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a, \end{cases}$$

where $a > 0$. In some books, $u(t-a) = H_a(t)$.

Example 20. *Compute the Laplace transform of the unit step function:*

$$\begin{aligned} L[u(t-a)] &= \int_0^\infty e^{-st} u(t-a)\, dt \\ &= \int_a^\infty e^{-st}\, dt \\ &= \left\{\frac{e^{-st}}{-s}\right\}_a^\infty \\ &= \frac{e^{-as}}{s}, \end{aligned}$$

for $s > 0$.

Assuming $f(t) = 0, t < 0$. The transform of a function $f(t), t \geq 0$, to $u(t-a)f(t-a)$ is called a shift of $f(t)$ by a units.

$$u(t-a)f(t-a) = \begin{cases} f(t-a) & \text{for } t \geq a \\ 0 & \text{for } t < a, \end{cases}$$

Theorem 9. *(The Second Shift Theorem) If F(s) exists for $s > c \geq 0$, then, for any constant $a \geq 0$,*

$$L\left\{u(t-a)f(t-a)\right\} = e^{-as}L\{f(t)\}.$$

Equivalently,

$$L^{-1}\left\{e^{-as}F(s)\right\} = u(t-a)\left[L^{-1}\left\{F(s)\right\}\right]_{t-a}.$$

Example 21.

- *If $f(t) = (t-1)^2u(t-1)$, then*

$$\mathcal{L}[f(t)] = e^{-s}\mathcal{L}[t^2] = \frac{2e^{-s}}{s^3}.$$

- *Let* $f(t) = \begin{cases} 0 & \textit{for } t < 0, \\ -1 & \textit{for } 0 \leq t \leq 2, \\ 1 & \textit{for } t > 2. \end{cases}$

  This can be expressed as $f(t) = -u(t) + 2u(t-2)$, so

$$\begin{aligned} \mathcal{L}[f(t)] &= -\mathcal{L}[u(t)] + 2\mathcal{L}[u(t-2)] \\ &= -\frac{1}{s} + \frac{2e^{-2s}}{s}. \end{aligned}$$

- *Let $f(t) = \sin(t)u(t-\pi)$. Then*

$$\mathcal{L}[f(t)] = e^{-\pi s}\mathcal{L}[\sin(t+\pi)] = e^{-\pi s}\mathcal{L}[-\sin(t)] = -e^{-\pi s}\frac{1}{s^2+1}.$$

If a function is defined as

$$f(t) = \begin{cases} f_1(t), & t \leq t_1; \\ f_2(t), & t_1 < t \leq t_2; \\ \cdots, & \cdots; \\ f_n(t), & t_{n-1} < t. \end{cases}$$

Then $f(t) = [1 - u(t-t_1)]f_1(t) + [u(t-t_1) - u(t-t_2)]f_2(t) + [u(t-t_2) - u(t-t_3)]f_3(t) + \cdots + [u(t-t_{n-2}) - u(t-t_{n-1})]f_{n-1}(t) + u(t-t_{n-1})f_n(t)$.

Example 22. *Solve the differential equation:*

$$f'' + f' - 6f = \begin{cases} 0 & t < 3 \\ 2 & 3 \le t < 5 \\ 4 & 5 \le t \end{cases}$$

with $f(0) = f'(0) = 0$.

Solution:

$$2u(t-3) - 2u(t-5) + 4u(t-5) = \begin{cases} 0 & t < 3 \\ 2 & 3 \le t < 5 \\ 4 & 5 \le t \end{cases}$$

The differential equation is

$$f'' + f' - 6f = 2u(t-3) + 2u(t-5)$$

with $f(0) = f'(0) = 0$ and we take the Laplace transform of both sides:

$$s^2F + sF - 6F = \frac{2}{s}\left(e^{-3s} + e^{-5s}\right)$$

or

$$F = \frac{2}{s(s+3)(s-2)}\left(e^{-3s} + e^{-5s}\right)$$

Now, partial fractions has

$$\frac{2}{s(s+3)(s-2)} = -\frac{1}{3s} + \frac{2}{15}\frac{1}{s+3} + \frac{1}{5}\frac{1}{s-2} = \mathcal{L}\left(-\frac{1}{3} + \frac{2}{15}e^{-3t} + \frac{1}{5}e^{2t}\right)$$

Now, in the expression for F this gets multiplied by various exponential factors, the effect of this is to delay the answer:

$$f = \left(-\frac{1}{3} + \frac{2}{15}e^{-3t+9} + \frac{1}{5}e^{2t-6}\right)u(t-3) + \left(-\frac{1}{3} + \frac{2}{15}e^{-3t+15} + \frac{1}{5}e^{2t-10}\right)u(t-5)$$

Example 23. *Solve the differential equation:*

$$f'' + 4f' + 7f = \begin{cases} 0 & t < 3 \\ 2 & 3 \le t < 5 \\ 4 & 5 \le t \end{cases}$$

with $f(0) = f'(0) = 0$.

Solution:

$$2u(t-3) - 2u(t-5) + 4u(t-5) = \begin{cases} 0 & t < 3 \\ 2 & 3 \le t < 5 \\ 4 & 5 \le t \end{cases}$$

The differential equation is

$$f'' + 4f' + 7f = 2u(t-3) + 2u(t-5)$$

with $f(0) = f'(0) = 0$ and we take the Laplace transform of both sides:

$$s^2F + 4sF + 7F = \frac{2}{s}\left(e^{-3s} + e^{-5s}\right)$$

or

$$F = \frac{2}{s(s^2+4s+7)}\left(e^{-3s} + e^{-5s}\right)$$

Now, partial fractions has

$$\begin{aligned}\frac{2}{s(s^2+4s+7)} &= \frac{1}{7s} - \frac{1}{7}\frac{s+2}{(s+2)^2+3} - \frac{1}{7}\frac{2}{(s+2)^2+3} \\ &= \mathcal{L}\left(\frac{1}{7} - \frac{1}{7}e^{-3t}\cos(\sqrt{3}t) - \frac{2}{7\sqrt{3}}e^{-3t}\sin(\sqrt{3}t)\right)\end{aligned}$$

Now, in the expression for F this gets multiplied by various exponential factors, the effect of this is to delay the answer.

1.4 CONVOLUTION AND THE DIRAC DELTA DISTRIBUTION

1.4.1 Convolutions

The *convolution* of $f(t)$ and $g(t)$ is defined by

$$(f * g) = \int_0^t f(u)g(t-u)\,du = \int_0^t f(t-u)g(u)\,du.$$

Remark. $f * g = g * f$.

Example 24. *Find the convolution $(f * g)(t)$ when $f(t) = t$, $g(t) = e^{2t}$ $(t \geq 0)$.*

Solution: From the definition of convolutions

$$\begin{aligned}
(f * g)(t) &= \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t \tau e^{2(t-\tau)}\,d\tau \\
&= \int_0^t \tau e^{2t}e^{-2\tau}\,d\tau = e^{2t}\int_0^t \tau e^{-2\tau}\,d\tau \\
\text{Use integration by parts with} \quad & u = \tau, \quad dv = e^{-2\tau}\,d\tau \\
& du = d\tau, \quad v = -\frac{1}{2}e^{-2\tau} \\
&= e^{2t}\int_0^t u\,dv = e^{2t}\left([uv]_0^t - \int_0^t v\,du\right) \\
&= e^{2t}\left(\left[-\frac{\tau}{2}e^{-2\tau}\right]_0^t - \int_0^t -\frac{1}{2}e^{-2\tau}\,d\tau\right) \\
&= e^{2t}\left(-\frac{t}{2}e^{-2t} + 0 + \frac{1}{2}\int_0^t e^{-2\tau}\,d\tau\right) \\
&= -\frac{t}{2} + \frac{e^{2t}}{2}\left[-\frac{1}{2}e^{-2\tau}\right]_0^t \\
&= -\frac{t}{2} + \frac{e^{2t}}{2}\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}\right) \\
&= -\frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{2t}
\end{aligned}$$

Convolution theorem

$$\mathcal{L}\{f * g(t)\} = F(s)G(s) = \mathcal{L}\{f\}\mathcal{L}\{g\}.$$

The LT of the convolution is the product of the LTs. (Or, equivalently, the inverse LT of the product is the convolution of the inverse LTs.)

Proof: Do a change-of-variables in the following double integral:

$$
\begin{aligned}
\mathcal{L}\{f * g(t)\} &= \int_0^\infty e^{-st} \int_0^t f(u)g(t-u)\,du\,dt \\
&= \int_0^\infty \int_u^\infty e^{-st} f(u)g(t-u)\,dt\,du \\
&= \int_0^\infty e^{-su} f(u) \int_u^\infty e^{-s(t-u)} g(t-u)\,dt\,du \\
&= \int_0^\infty e^{-su} f(u)\,du \int_0^\infty e^{-sv} g(v)\,dv \\
&= \mathcal{L}\{f\}\mathcal{L}\{g\}.
\end{aligned}
$$

Example 25. *Calculate the inverse Laplace transform of* $\frac{1}{s^3-s^2}$.

Solution: This can be computed using partial fractions and LT tables. However, it can also be computed using convolutions.

First we factor the denominator, as follows

$$
\frac{1}{s^3 - s^2} = \frac{1}{s^2}\frac{1}{s-1}.
$$

We know the inverse Laplace transforms of each term:

$$
\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t, \qquad \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = e^t
$$

We apply the convolution theorem:

$$
\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-1}\right] &= \int_0^t u e^{t-u}\,du \\
&= e^t\left[-ue^{-u}\right]_0^t - e^t \int_0^t -e^{-u}\,du \\
&= -t - 1 + e^t
\end{aligned}
$$

Therefore,

$$
\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-1}\right](t) = e^t - t - 1.
$$

Example 26. *Find the convolution*

$$
f(t) = 1 * 2 * 3 * 4 * 5.
$$

Solution: Take the LT. Since the LT of the convolution is the product of the LTs:

$$\mathcal{L}\{1 * 2 * 3 * 4 * 5\} = 5!(1/s)^5 = \frac{5!}{s^5} = F(s).$$

We know from LT tables that $\mathcal{L}^{-1}\left[\frac{4!}{s^5}\right](t) = t^4$, so

$$f(t) = \mathcal{L}^{-1}[F(s)](t) = 5\mathcal{L}^{-1}\left[\frac{4!}{s^5}\right](t) = 5t^4.$$

1.4.2 Dirac delta function

Consider the function

$$\delta_k(t) = \begin{cases} k, & 0 < t \leq \frac{1}{k}; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\delta_k(t-a) = \begin{cases} k, & a < t \leq a + \frac{1}{k}; \\ 0, & \text{otherwise.} \end{cases}$$

Properties:

- $\delta_k(t-a) = ku(t-a) - ku(t-(a+\frac{1}{k}))$.
- $\int_{-\infty}^{\infty} \delta_k(t-a)dt = 1$.
- $\int_0^{\infty} \delta_k(t-a)f(t)dt = \int_a^{a+1/k} kf(t)dt = f(t^*)$, where t^* is a number between a and $a + \frac{1}{k}$.
- $L\{\delta_k(t-a)\} = e^{-as}\frac{k}{s}(1 - e^{-s/k})$.
- $\lim_{k\to\infty} L\{\delta_k(t-a)\} = e^{-as}$.

Definition 1. *Dirac delta distribution, or Dirac delta (generalized) function, is defined as*

$$\delta(t-a) = \lim_{k\to\infty} \delta_k(t-a).$$

The "Dirac delta function" $\delta(t)$ is technically not a function. Roughly speaking, it may be thought of as being analogous to a radar "ping": if a continuous function $f(t)$ represents an objects' trajectory then the delta function "samples" its value at $t = 0$.

Properties:

- $L\{\delta(t)\} = 1$.
- $L\{\delta(t-a)\} = e^{-as}$.

- $\delta_a(f) = \int_0^\infty \delta(t-a) f(t) dt = f(a)$, which is a linear functional.
- If $a > 0$ then $\int_0^\infty \delta(t-a) f(t) e^{-st}\, dt = f(a) e^{-as}$.

Example 27. *Solve* $x'' + x = -\delta(t-\pi)$, $x(0) = x'(0) = 0$.

Remark. This models a unit mass attached to an undamped spring suspended from a board with no initial displacement or initial velocity. At time $t\pi$, the board is hit very hard (say with a hammer blow) in the upward direction. As we will see, this will start the mass to oscillate.

Solution: Taking Laplace transforms gives $s^2 X(s) + X(s) = e^{-\pi s}$, so

$$X(s) = -\frac{1}{s^2+1} e^{-\pi s}.$$

The inverse LT is $x(t) = \sin(t) u(t-\pi)$.

Example 28. *Solve*

$$f'' + f' - 6f = \delta(t-4)$$

with $f(0) = f'(0) = 0$.

Solution: Using $\mathcal{L}[\delta(t-a)] = e^{as}$ this gives

$$s^2 F + sF - 6F = e^{-4s}$$

or

$$F = \frac{e^{-4s}}{(s+3)(s-2)}$$

By partial fractions we have

$$\frac{1}{(s+3)(s-2)} = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{1}{s+3} = \mathcal{L}\left(\frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}\right)$$

so, the e^{-4s} causes a delay of four and we have

$$f = \left(\frac{1}{5}e^{2t-8} - \frac{1}{5}e^{-3t+12}\right) u(t-4)$$

2 II Series Solutions of ODEs

We have fully investigated solving second order linear differential equations with constant coefficients:

$$Ay'' + By' + Cy = 0,$$

where A,B,C are constants. Now we will explore how to find solutions to second order linear differential equations whose coefficients are not necessarily constant:

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

2.1 Taylor Series

Definition 2. *The Taylor series about x_0 of a function $f(x)$ is the series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

*There exists $R \geq 0$ such that the series is convergent in $|x - x_0| < R$ and divergent in $|x - x_0| > R$. The number R is called **Radius of Convergence**. We have*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \quad |x - x_0| < R.$$

Remark. We use Ratio Test to find R.

Example 29. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,\ R = 1.$

$\frac{1}{3-2x} = \frac{1}{1-2(x-1)} = \sum_{n=0}^{\infty} 2^n (x-1)^n.\ R = \frac{1}{2}.$

$\frac{1}{3-2x} = \frac{1}{3}\frac{1}{1-\frac{2}{3}x} = \frac{1}{3}\sum_{n=0}^{\infty} (\frac{2}{3})^n x^n.\ R = \frac{3}{2}.$

2.2 Analytic functions

Definition 3. *A function $f(x)$ is analytic at x_0 if f has Taylor series about x_0 which converges to $f(x)$ in an interval containing x_0.*

Example 30. *$f(x) = \frac{1}{1-x}$ is analytic at $x = 0$; $f(x) = \sqrt{x}$ is not analytic at 0, since $f'(0)$ does not exist; $f(x) = e^x$ is analytic at any x.*

Remark: If f and g are analytic at x_0, then cf, $f \pm g$, fg, f/g (if $g(x_0) \neq 0$) are analytic at x_0.
Remark: If f is analytic at x_0, then its Taylor series about x_0 is unique.

2.3 Singular point and ordinary point

Consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

If we divide two sides by $P(x)$, then the equation is changed to

$$y'' + p(x)y' + q(x)y = 0. \qquad (1)$$

Definition 4. *If both $p(x)$ and $q(x)$ are analytic at a point x_0, then x_0 is called an ordinary point. Otherwise, it is called a singular point.*

Remark. If $p(x)$ and $q(x)$ are polynomials, then any point is an ordinary point.

Example 31. *The following equation has singular points $x = 1, 2$:*

$$y'' + \frac{x + e^x}{x - 1}y' + \frac{x + 1}{x - 2}y = 0.$$

Theorem 10. *If x_0 is an ordinary point of the ODE (1), then the general solution of (1) is analytic at x_0, and is therefore given by*

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

in an open interval containing x_0 with two arbitrary coefficients (usually a_0 and a_1). The radius R is at least the distance from x_0 to the closest singular point of the equation.

Example 32. *Consider the following equation with two singular points $x = 1, 2$:*

$$y'' + \frac{x + e^x}{x - 1}y' + \frac{x + 1}{x - 2}y = 0.$$

Then the series solution about $x_0 = 0$ has $R \geq 1$; the series solution about $x_0 = \frac{3}{4}$ has $R \geq \frac{1}{4}$.

Example 33. *Consider the following equation:*

$$y'' + \frac{1}{x^2 + 1}y' + xy = 0.$$

Then the series solution about $x_0 = 0$ has $R \geq 1$; the series solution about $x_0 = 2$ has $R \geq \sqrt{5}$.

Definition 5. *Let x_0 be a singular point. If both $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ are analytic at x_0, then x_0 is called a* ***regular singular point*** *(non-essential singular point). If at least one of $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ is not analytic at x_0, then x_0 is called an* ***irregular singular point*** *(essential singular point).*

Example 34. *Find all singularities and classify them:*
(1) $y'' + \frac{x-1}{x(x-2)}y' + \frac{x}{x^2(x-2)^3}y = 0.$
(2) $(x\sin x)y'' + (\cos x)y' + e^x y = 0.$

Solution: (1) It has one regular singular point $x = 0$ and one irregular singular point $x = 2$.

(2) We change the equation to

$$y'' + \frac{\cos x}{x\sin x}y' + \frac{e^x}{x\sin x}y = 0.$$

Note that $\frac{\cos x}{x\sin x}$ and $\frac{\cos x}{x\sin x}$ are not analytic at zeros of the denominator $x\sin x$. Let $x\sin x = 0$, we imply that $x = k\pi$, $k = 0, \pm1, \pm2, ...$They are all singular points, in which $x = 0$ is irregular, and others are regular.

2.4 Power Series Solution

Now we solve an equation at ordinary points.

Example 35. *Let $y = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots$ be the series solution about $x = 0$ of the initial value problem: $y''-2xy'+y = 0, \quad y(0) = -2, y'(0) = -1$. Find a_0, a_1, a_2, a_3, a_4.*

Solution: From $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$ we imply that

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots, \quad y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots.$$

Substitute them into the differential equation,

$$(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots) - 2x(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots)$$
$$+(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots) = 0,$$

i.e.,

$$(a_0 + 2a_2) + (-a_1 + 6a_3)x + (12a_4 - 3a_2)x^2 \cdots = 0.$$

We have

$$a_0 = y(0) = -2,$$

$$a_1 = y'(0) = -1,$$

$$a_0 + 2a_2 = 0, \Rightarrow a_2 = 1$$

$$-a_1 + 6a_3 = 0, \Rightarrow a_3 = -1/6.$$

$$12a_4 - 3a_2 = 0 \Rightarrow a_4 = 1/4.$$

Example 36. *Consider the DE:* $y'' - 2xy' + y = 0$. *Note that* $x_0 = 0$ *is a regular point. So we shall attempt to find a series solution in the form:*

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

1) Find the recursive relation of the coefficients in the series solution about $x_0 = 0$.
2) Solve the recursive relation.
3) Find the particular solution with $y(0) = 1, y'(0) = 2$.

Solution: 1) From $y = \sum_{n=0}^{\infty} a_n x^n$ we have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \Rightarrow xy' = \sum_{n=1}^{\infty} n a_n x^n,$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_n x^n.$$

Substitute all of them into the DE, we imply that

$$a_{n+2} = \frac{2n-1}{(n+1)(n+2)} a_n.$$

2) From the recursive relation above,

$$a_{2n} = \frac{3 \cdots 7 \cdot 11 \cdots (4n-5)}{(2n)!} a_0,$$

$$a_{2n+1} = \frac{1 \cdots 5 \cdot 9 \cdots (4n-3)}{(2n+1)!} a_1.$$

We have

$$y = a_0\left(1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty}\frac{3\cdots 7\cdot 11\cdots(4n-5)}{(2n)!}x^{2n}\right)$$

$$+a_1\left(x + \sum_{n=1}^{\infty}\frac{1\cdots 5\cdot 9\cdots(4n-3)}{(2n+1)!}x^{2n+1}\right)$$

3) From $y(0) = 1$ we have $a_0 = 1$; by $y'(0) = 2$, we get $a_1 = 2$. Thus

$$y = \left(1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty}\frac{3\cdots 7\cdot 11\cdots(4n-5)}{(2n)!}x^{2n}\right)$$

$$+2\left(x + \sum_{n=1}^{\infty}\frac{1\cdots 5\cdot 9\cdots(4n-3)}{(2n+1)!}x^{2n+1}\right)$$

Example 37. *Consider the DE: $y'' - 2xy' + sy = 0$. Note that $x_0 = 0$ is a regular point. So we shall attempt to find a series solution in the form:*

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

1) Find the recursive relation of the coefficients in the series solution about $x_0 = 0$.
2) Solve the recursive relation.

Solution: 1)

$$y = \sum_{n=0}^{\infty}[(s-2n)a_n + (n+1)(n+2)a_{n+2}]x^n = 0.$$

2)

$$a_{2n+2} = \frac{(-1)^{n+1}s(s-4)(s-8)...(s-4n)}{(2n+2)!}a_0, \quad a_{2n+3} = \frac{(-1)^{n+1}(s-2)(s-6)...(s-4n-2)}{(2n+3)!}a_1.$$

2.5 Cauchy-Euler Equations

An equation of the form

$$a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + ... + a_1 xy' + a_0 y = 0,$$

is called a *Cauchy-Euler* equation.

Theorem: Second-order Cauchy-Euler equation:

$$a_2 x^2 y'' + a_1 xy' + a_0 y = 0,$$

with $a_2,\ a_1$ and a_0 constants. In standard form, the equation is

$$y'' + \frac{A}{x}y' + \frac{B}{x^2}y = 0.$$

The auxiliary equation is

$$a_2 r\,(r-1) + a_1 r + a_0 = 0, \quad or, \quad a_2 r^2 + (a_1 - a_2)\,r + a_0 = 0.$$

Let r_1 and r_2 be the two solutions of the auxiliary equation.

(i) If $r_1 \neq r_2$ are real, then $y_1 = |x|^{r_1}$ and $y_2 = |x|^{r_2}$.

(ii) If $r_1 = r_2$ (real), then $y_1 = |x|^{r_1}$ and $y_2 = x^{r_1} \ln|x|$.

(iii) If $r_1, r_2 = \alpha \pm i\beta$ (complex), then $y_1 = |x|^{\alpha}\cos(\beta \ln|x|)$ and $y_2 = |x|^{\alpha}\sin(\beta \ln|x|)$.

Proof. Since $p(x) = \dfrac{A}{x}$ and $q(x) = \dfrac{B}{x^2}$ are undefined at $x = 0$, the solution may be undefined at $x = 0$. Thus, we assume that $x \neq 0$. A Cauchy-Euler equation can be transformed into a constant-coefficient equation as follows:

For $x > 0$, let $x = e^t$ and $y(x) = z(t)$. Then $t = \ln(x)$ and, by the chain rule,

$$\frac{dy}{dx} = \frac{dz}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dz}{dt}, \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dz}{dt} + \frac{1}{x}\frac{d^2z}{dt^2}\frac{dt}{dx} = -\frac{1}{x^2}\frac{dz}{dt} + \frac{1}{x^2}\frac{d^2z}{dt^2},$$

and the equation $x^2y'' + Axy' + By = 0$ becomes $\left[\dfrac{d^2z}{dt^2} - \dfrac{dz}{dt}\right] + A\dfrac{dz}{dt} + Bz = 0$, or

$$z'' + (A-1)z' + Bz = 0,$$

which has constant coefficients.

If $z_1(t)$ and $z_2(t)$ are two independent solutions of $z'' + (A-1)z' + Bz = 0$, then two independent solutions of $x^2y'' + Axy' + By = 0$ are given by

$$y_1(x) = z_1(\ln x) \ \text{ and } \ y_2(x) = z_2(\ln x).$$

Since solutions of a constant-coefficient equation are sought in the form $z = e^{rt}$ and $y(x) = z(t)$ with $t = \ln(x)$, $y(x) = e^{rt} = e^{r\ln(x)} = e^{\ln(x^r)} = x^r$. Thus, solutions of an Euler equation can be sought directly in the form $y = x^r$.

If $r_1 \neq r_2$ are real, then $z_1 = e^{r_1 t}$ and $z_2 = e^{r_2 t} \to y_1 = x^{r_1}$ and $y_2 = x^{r_2}$.

If $r_1 = r_2$ (real), then $z_1 = e^{r_1 t}$ and $z_2 = te^{r_1 t} \to y_1 = x^{r_1}$ and $y_2 = x^{r_1} \ln(x)$.

If $r_1, r_2 = \alpha \pm i\beta$ (complex), then $z_1 = e^{\alpha t} \cos(\beta t)$ and $z_2 = e^{\alpha t} \sin(\beta t) \to y_1 = x^{\alpha} \cos[\beta \ln(x)]$ and $y_2 = x^{\alpha} \sin[\beta \ln(x)]$.

For $x < 0$, let $x = -e^t$ and $y(x) = z(t)$. Then $t = \ln(-x)$, and the same equation for $z(t)$ results. In either case, $t = \ln|x|$.

Since $|x| = \left\{ \begin{array}{rl} x, & x > 0 \\ -x, & x < 0 \end{array} \right\}$, replacing x by $|x|$ gives the solutions for any $x \neq 0$. Thus,

If $r_1 \neq r_2$ are real, then $y_1 = |x|^{r_1}$ and $y_2 = |x|^{r_2}$.

If $r_1 = r_2$ (real), then $y_1 = |x|^{r_1}$ and $y_2 = x^{r_1} \ln|x|$.

If $r_1, r_2 = \alpha \pm i\beta$ (complex), then $y_1 = |x|^{\alpha} \cos(\beta \ln|x|)$ and $y_2 = |x|^{\alpha} \sin(\beta \ln|x|)$.

Example 38. *Solve the following equations:*

1. $x^2 y'' + 2xy' - 2y = 0, \ x > 0.$
   Solution: This is an Euler equation. The auxiliary equation is
   $$r^2 + r - 2 = 0 \ \Rightarrow \ (r+2)(r-1) = 0 \ \Rightarrow y = c_1 x^{-2} + c_2 x = \frac{c_1}{x^2} + c_2 x.$$

2. $x^2 y'' + 5xy' + 4y = 0, \ x > 0.$

   Solution: This is an Euler equation. The auxiliary equation is
   $$r^2 + 4r + 4 = 0 \ \Rightarrow \ (r+2)^2 = 0 \ \Rightarrow \ y = \frac{c_1}{x^2} + c_2 \frac{\ln(x)}{x^2}.$$

3. $x^2 y'' + 4xy' + 4y = 0, \ x > 0.$

   Solution: This is an Euler equation. The auxiliary equation is
   $$r^2 + 3r + 4 = 0 \ \Rightarrow \ r = \frac{-3 \pm \sqrt{9 - 16}}{2} = -\frac{3}{2} \pm \frac{\sqrt{7}}{2} i \ \Rightarrow$$

$$y = x^{-\frac{3}{2}}\left[c_1\cos\left(\frac{\sqrt{7}}{2}\ln(x)\right) + c_2\sin\left(\frac{\sqrt{7}}{2}\ln(x)\right)\right].$$

4. $3(x-5)^2y'' + 6(x-5)y' - 6y = 0,\ x \neq 5.$

   Solution: This is an Euler equation. The auxiliary equation is

$$r^2 + r - 2 = 0 \ \Rightarrow\ (r+2)(r-1) = 0 \ \Rightarrow y = c_1|x-5|^{-2} + c_2|x-5|.$$

5. $x^2y'' - 6y = x^3\ln(x),\ x > 0.$

   Solution: This is an Euler equation. The auxiliary equation is

$$r^2 - r - 6 = 0 \ \Rightarrow\ (r-3)(r+2) = 0 \ \Rightarrow\ y_1 = x^3,\ y_2 = x^{-2},\ y_h = c_1x^3 + c_2x^{-2}.$$

$f(x) = \dfrac{x^3\ln(x)}{x^2} = x\ln(x),$

$W[y_1, y_2] = \begin{vmatrix} x^3 & x^{-2} \\ 3x^2 & -2x^{-3} \end{vmatrix} = -5,\ u_1 = -\displaystyle\int \frac{y_2 f}{W}\,dx = \frac{1}{5}\int \frac{\ln(x)}{x}\,dx = \frac{1}{10}[\ln(x)]^2,$

$u_2 = \displaystyle\int \frac{y_1 f}{W}\,dx = -\frac{1}{5}\int x^4\ln(x)\,dx = -\frac{1}{25}x^5\ln(x) + \frac{1}{25}\int x^4\,dx =$

$-\dfrac{1}{25}x^5\ln(x) + \dfrac{1}{125}x^5 \ \Rightarrow$

$y_p = u_1y_1 + u_2y_2 = \dfrac{1}{10}[\ln(x)]^2x^3 + \left[-\dfrac{1}{25}x^5\ln(x) + \dfrac{1}{125}x^5\right]x^{-2} =$

$\dfrac{1}{10}[\ln(x)]^2x^3 - \dfrac{1}{25}x^3\ln(x) + \dfrac{1}{125}x^3 \ \Rightarrow$

$y = y_p + y_h = \dfrac{1}{10}[\ln(x)]^2x^3 - \dfrac{1}{25}x^3\ln(x) + \dfrac{1}{125}x^3 + c_1x^3 + c_2x^{-2}.$

Theorem: Third-order Cauchy-Euler equation:

$$a_3x^3y''' + a_2x^2y'' + a_1xy' + a_0y = 0$$

Let $y = x^m$. Similarly, we have

$$y' = mx^{m-1},\ y'' = m(m-1)x^{m-2},\ y''' = m(m-1)(m-2)x^{m-3}$$

Then

$$a_3x^3m(m-1)(m-2)x^{m-3} + a_2x^2m(m-1)x^{m-2} + a_1xmx^{m-1} + a_0x^m = 0$$

$$x^m\left(a_3m(m-1)(m-2) + a_2m(m-1) + a_1m + a_0\right) = 0$$

Thus we get the auxiliary eqn:

$$a_3m(m-1)(m-2) + a_2m(m-1) + a_1m + a_0 = 0.$$

Example 39. *Solve the Cauchy-Euler equation*

$$(a)\ x^3y''' + xy' - y = 0 \qquad (b)\ x^3y''' + x^2y'' - 2xy' + 2y = 0$$

1. $m^3 + (0-3)\,m^2 + (2-0+1)\,m - 1 = m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0,$

   $m = 1, 1, 1, \ \ y_1 = x,\ y_2 = x\ln x,\ y_3 = x\,(\ln x)^2$

   $y_c = c_1x + c_2x\ln x + c_3x\,(\ln x)^2$

2. $m^3 + (1-3)\,m^2 + (2-1-2)\,m + 2 = m^3 - 2m^2 - m + 2 = (m-1)\,(m-2)\,(m+1) = 0$

   $m = 1,\ 2,\ -1,\ y_1 = x,\ y_2 = x^2,\ y_3 = x^{-1}$

   $y_c = c_1x + c_2x^2 + c_3\dfrac{1}{x}$

2.6 The Frobenius Method

Frobenius method is important for equations with coefficients that have singularities, so that power series method can no longer handle them.

Theorem 6. Let $x_0 = 0$ be a regular singular point of the DE

$$y'' + p(x)y' + q(x)y = 0$$

with

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Let

$$y = x^r \sum_{n=0}^{\infty} c_n(r) x^n = \sum_{n=0}^{\infty} c_n(r) x^{n+r}, \quad c_0(r) = 1.$$

Let r_1 and r_2 (where $r_1 \geq r_2$ if both are real) be two roots of the indicial equation

$$r^2 + (p_0 - 1)r + q_0 = 0.$$

Case (i): If $r_1 - r_2$ is not an integer, then the two linearly independent solutions are given by:

$$y_1 = \sum_{n=0}^{\infty} c_n(r_1) x^{n+r_1}, \quad y_2 = \sum_{n=0}^{\infty} c_n(r_2) x^{n+r_2}, x > 0.$$

Case (ii): If $r_1 = r_2$, then the two linearly independent solutions are given by:

$$y_1 = \sum_{n=0}^{\infty} c_n(r_1) x^{n+r_1}, \quad y_2 = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r_1}, x > 0,$$

where $b_n = c_n'(r_1)$.

Case (iii): If $r_1 - r_2 = N$ is a positive integer, then the two linearly independent solutions are given by:

$$y_1 = \sum_{n=0}^{\infty} c_n(r_1) x^{n+r_1}, \quad y_2 = Ay_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2}, x > 0,$$

where

$$A = \lim_{r \to r_2} [(r - r_2) c_N(r)], \quad b_n = \frac{d}{dr} [(r - r_2) c_n(r)]_{r=r_2}.$$

The radius of convergence of every one of the above is at least as great as the distance from $x_0 = 0$ to the nearest other singular point of the equation.

Example 40. *Find the general solution of*

$$2x^2y'' + (x + 2x^2)y' - y = 0$$

for $x > 0$ *near* $x_0 = 0$.

Solution: Step 1: Determine whether $x_0 = 0$ is an ordinary point or a regular singular point. Write the equation as

$$y'' + (\frac{1}{2x} + 1)y' - \frac{1}{2x^2}y = 0,$$

We have

$$xp(x) = \frac{1}{2} + x, x^2q(x) = -\frac{1}{2}.$$

So $x_0 = 0$ is a regular singular point.

Step 2: Find and solve the indicial equation. Note that $p_0 = \frac{1}{2}, q_0 = -\frac{1}{2}$. The indicial equation is:

$$r^2 + (p_0 - 1)r + q_0 = 0. \Rightarrow r^2 - \frac{1}{2}r - \frac{1}{2} = 0. \Rightarrow r_1 = 1, r_2 = -\frac{1}{2}.$$

Note that $r_1 - r_2 = 1.5$, so we have Case (i).

Step 3: Find the recursive relation about $c_n(r)$. Let

$$y = \sum_{n=0}^{\infty} c_n(r)x^{n+r}, \quad c_0(r) = 1. \Rightarrow$$

$$y' = \sum_{n=0}^{\infty}(n + r)c_n(r)x^{n+r-1}, y'' = \sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n(r)x^{n+r-2}.$$

Substitute them into (5.1) we have

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(r)x^{n+r-2} + (x+2x^2)\sum_{n=0}^{\infty}(n+r)c_n(r)x^{n+r-1} - \sum_{n=0}^{\infty}c_n(r)x^{n+r} = 0, \Rightarrow$$

$$\sum_{n=0}^{\infty}2(n + r)(n + r - 1)c_n(r)x^{n+r} + \sum_{n=0}^{\infty}(n + r)c_n(r)x^{n+r}$$
$$+\sum_{n=0}^{\infty}2(n + r)c_n(r)x^{n+r+1} - \sum_{n=0}^{\infty}c_n(r)x^{n+r} = 0, \Rightarrow$$

$$\sum_{n=0}^{\infty}[2(n + r)(n + r - 1) + (n + r) - 1]c_n(r)x^{n+r} + \sum_{n=0}^{\infty}2(n + r)c_n(r)x^{n+r+1} = 0, \Rightarrow$$

$$\sum_{n=0}^{\infty}[(2n+2r+1)(n+r-1)]c_n(r)x^{n+r} + \sum_{n=1}^{\infty} 2(n-1+r)c_{n-1}(r)x^{n+r} = 0, \Rightarrow$$

$$(2r+1)(r-1)]c_0(r)x^r + \sum_{n=1}^{\infty}\{[(2n+2r+1)(n+r-1)]c_n(r) + 2(n-1+r)c_{n-1}(r)\}x^{n+r} = 0, \Rightarrow$$

$$(2r+1)(r-1)]c_0(r) = 0, \quad \{[(2n+2r+1)(n+r-1)]c_n(r) + 2(n-1+r)c_{n-1}(r)\} = 0, n \geq 1.$$

Since $c_0(r) = 1 \neq 0$, the above first equation results in our indicial equation $(2r+1)(r-1) = 0$. The second equation gives

$$c_n(r) = \frac{-2}{2n+2r+1}c_{n-1}(r), \quad n \geq 1. \tag{10}$$

Step 4: Find y_1. Take $r = r_1 = 1$, by (10),

$$c_n(1) = \frac{-2}{2n+3}c_{n-1}(1), n \geq 1. \Rightarrow$$

$$c_1(1) = \frac{-2}{5}c_0(1) = \frac{-2}{5},$$

$$c_2(1) = \frac{-2}{7}c_1(1) = \frac{(-2)^2}{5(7)}, ...,$$

$$c_n(1) = \frac{(-2)^n}{5(7)...(2n+3)}, n \geq 1.$$

Therefore,

$$y_1 = x + \sum_{n=1}^{\infty}\frac{(-2)^n}{5(7)...(2n+3)}x^{n+1} = x^{r_1}\left(1 + \sum_{n=1}^{\infty}\frac{(-2)^n}{5(7)...(2n+3)}x^n\right).$$

Step 5: Find y_2. Take $r = r_2 = -\frac{1}{2}$, by (10),

$$c_n(-\frac{1}{2}) = \frac{-2}{2n}c_{n-1}(-\frac{1}{2}) = \frac{-1}{n}c_{n-1}(-\frac{1}{2}), n \geq 1. \Rightarrow$$

$$c_1(-\frac{1}{2}) = \frac{-1}{1}c_0(-\frac{1}{2}) = -1,$$

$$c_2(-\frac{1}{2}) = \frac{-1}{2}c_1(-\frac{1}{2}) = \frac{(-2)^2}{2}, ...,$$

$$c_n(-\frac{1}{2}) = \frac{(-1)^n}{n!}, n \geq 1.$$

Therefore,

$$y_2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{n-\frac{1}{2}} = x^{-\frac{1}{2}}e^{-x}.$$

Step 6: The general solution is $y(x) = c_1 y_1 + c_2 y_2$, where c_1 and c_2 are constants.
Remark. To get solutions for $x < 0$ or for $x \neq 0$ near $x_0 = 0$, replace x^{r_1}, x^{r_2} and $\ln x$ in Theorem 6 by $|x|^{r_1}$, $|x|^{r_2}$ and $\ln |x|$.
Remark. If $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are not polynomials but are given by Taylor series, then it's possible to obtain approximate solutions.

Example 41. *Solve Cauchy-Euler equation*

$$x^2 y'' + 2xy' + \frac{3}{16} y = 0, \quad x > 0$$

The indicial equation is:

$$r^2 + r + \frac{3}{16} = 0.$$

The two solutions are $r_1 = -\frac{1}{4}$, $r_2 = -\frac{3}{4}$, $r_1 - r_2 = \frac{1}{2}$. We get a basis $y_1 = x^{-1/4}$, $y_2 = x^{-3/4}$.

Example 42. *Solve Hyper-geometric differential equation*

$$x(x-1)y'' + (3x-1)y' + y = 0, \quad x > 0. \tag{11}$$

Note that $p(x) = (3x-1)/(x-1)$ and $q(x) = x/(x-1)$. We have $b_0 = b(0) = 1$, $c_0 = c(0) = 0$. Thus the indicial equation is $r^2 = 0$, which has double root $r = 0$. We have Case (ii). Substitute

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n(r) x^{n+r}, \quad c_0(r) = 1 \\ y' &= \sum_{n=0}^{\infty} (n+r) c_n(r) x^{n+r-1}, \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n(r) x^{n+r-2} \end{aligned}$$

into (11), we have

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} 3(n+r) c_n x^{n+r}$$

$$- \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0,$$

which implies that

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1)c_n - (n+r+1)(n+r)c_{n+1} + 3(n+r)c_n - (n+r+1)c_{n+1} + c_n] x^{n+r} - r^2 c_0 x^{r-1} = 0.$$

Simplifying this and using indicial equation $r^2 = 0$, we get

$$\sum_{n=0}^{\infty}(n+r+1)^2(c_n - c_{n+1})x^{n+r} = 0.$$

Thus $c_{n+1} = c_n = \cdots = 1$. So

$$y_1(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

To get another independent solution, note that $b_n = c_n'(r_1) = 0$ for all n,

$$y_2 = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r} = \frac{\ln x}{1-x}.$$

Example 43. *(Case III) Solve*

$$(x^2 - x)y'' - xy' + y = 0, \quad x > 0. \tag{12}$$

Solution: The indicial equation is $r(r-1) = 0$, the roots are $r_1 = 1$ and $r_2 = 0$. This is the Case III with $N = r_1 - r_2 = 1$.

Let $y = \sum_{n=0}^{\infty} c_n x^{n+1}$. Substitute this into (12),

$$(x^2 - x)\sum_{n=0}^{\infty}(n+1)(n+1-1)c_n x^{n-1} - x\sum_{n=0}^{\infty}(n+1)c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

Collecting like terms

$$\sum_{n=0}^{\infty}\left[n^2 c_n - (n+2)(n+1)c_{n+1}\right] x^{n+1} = 0.$$

We derive

$$c_{n+1} = \frac{n^2}{(n+2)(n+1)} c_n \quad (n = 0, 1, ...), \Rightarrow$$

$c_1 = c_2 = \cdots = 0$. We get a solution $y_1(x) = c_0 x = x$.

To get a second solution, we let

$$y_2 = k y_1 \ln(x) + \sum_{n=0}^{\infty} d_n x^{n+r_2} = kx\ln(x) + \sum_{n=0}^{\infty} d_n x^n,$$

where k is a constant. Substitute y_2 into the differential equation,

$$-k + \sum_{n=0}^{\infty}[d_n(n-1)^2 - d_{n+1}(n+1)n]x^n = 0, \Rightarrow$$

$$d_0 = k, d_2 = d_3 = ... = 0.$$

Thus

$$y_2 = kx\ln(x) + k + d_1 x.$$

2.7 Bessel's Equation

Definition 6. *The DE*

$$x^2y''(x) + xy'(x) + (\lambda^2x^2 - \nu^2)y(x) = 0$$

is called Bessel's equation of order ν with parameter λ, where $\nu \geq 0$ and λ are constants. If $\lambda = 1$, then it is called Bessel's equation of order ν.

If $\lambda = 0$, it is Euler equation. If $\lambda \neq 0$, by substitution $z = \lambda x$ and $w(z) = y(x)$, the equation above can be changed to

$$z^2w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = 0.$$

Definition 7.

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

is called the Bessel function of the first kind of order ν; For any $\nu \geq 0$,

$$Y_\nu(x) = \lim_{\mu \to \nu} \frac{\cos(\pi\mu)J_\mu(x) - J_{-\mu}(x)}{\sin(\pi\mu)}$$

is called the Bessel function of the second kind of order ν.

Theorem For any $x > 0$, two linearly independent solutions $y_1(x)$ and $y_2(x)$ of

$$x^2y''(x) + xy'(x) + (\lambda^2x^2 - \nu^2)y(x) = 0$$

are

$$y_1(x) = J_\nu(\lambda x), \quad y_2(x) = \begin{cases} J_{-\nu}(\lambda x), & \text{if } \nu > 0 \text{ is not an integer;} \\ Y_\nu(\lambda x), & \text{for any } \nu \geq 0. \end{cases}$$

The general solution is $y(x) = c_1y_1 + c_2y_2$, where c_1 and c_2 are constants.

Remark. If $\nu > 0$ is an integer, then $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$.

Example 44. *Find two linearly independent solutions, valid for $x > 0$:*

(1)

$$x^2y''(x) + xy'(x) + (4x^2 - 9)y(x) = 0.$$

Solution: *Note that $\lambda^2 = 4$ and $\nu^2 = 9$, $\Rightarrow \lambda = 2$ and $\nu = 3$. Hence*

$$y_1(x) = J_3(2x), \quad y_2(x) = Y_3(2x).$$

(2)

$$x^2y''(x) + xy'(x) + (4x^2 - 0.25)y(x) = 0.$$

Solution: *Note that* $\lambda^2 = 4$ *and* $\nu^2 = 0.25$, $\Rightarrow \lambda = 2$ *and* $\nu = 0.5$. *Since* ν *is not an integer,*

$$y_1(x) = J_{0.5}(2x), \quad y_2(x) = J_{-0.5}(2x).$$

(3)

$$xy''(x) + y'(x) + 0.04xy(x) = 0.$$

Solution: *This not Bessel equation. However, by multiplying two sides with* x, *we get*

$$x^2y''(x) + xy'(x) + 0.04x^2y(x) = 0$$

which is Bessel equation with $\lambda^2 = 0.04$ *and* $\nu^2 = 0$, $\Rightarrow \lambda = 0.2$ *and* $\nu = 0$. *Since* ν *is an integer,*

$$y_1(x) = J_0(0.2x), \quad y_2(x) = Y_0(0.2x).$$

Remark. If we need solution for $x \neq 0$, then

$$J_\nu(x) = \left(\frac{|x|}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k}$$

3 III Fourier Series

3.1 Pre-knowledge

1. Trig Identities:

$$\sin(n\pi + \frac{\pi}{2}) = (-1)^n; \quad \cos(n\pi) = (-1)^n, \quad n \text{ is an integer.}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}.$$

$$\sin a \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}, \quad \cos a \cos b = \frac{\cos(a+b) + \cos(a-b)}{2}.$$

$$\sin a \cos b = \frac{\sin(a+b) + \sin(a-b)}{2}.$$

2. A function $f(x)$ is called $p-$*periodic* if $p > 0$ is the smallest number such that $f(x+p) = f(x)$ for any x. The number p is called the period. For example, $\cos kx$ and $\sin kx$ are $\frac{2\pi}{k}$-periodic.

3. Odd-Even function: If $f(-x) = -f(x)$ for all $x \in [-a, a]$, then $f(x)$ is odd on $[-a, a]$; If $f(-x) = f(x)$ for all $x \in [-a, a]$, then $f(x)$ is even on $[-a, a]$. For example, $\sin kx$ is odd, $\cos kx$ is even.

   - If $f(x)$ is odd on $[-a, a]$, then $\int_{-a}^{a} f(x)dx = 0$.
   - If $f(x)$ is even on $[-a, a]$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

4. If m, n are non-negative integers, and $m \neq n$, then

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0, \quad \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0.$$

   Also, even if we drop the restriction $m \neq n$,

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0.$$

5. If n is a positive integer, then

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi.$$

6. Integration by parts: $\int_a^b udv = uv|_a^b - \int_a^b vdu$.

7. Integration by substitution: $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

3.2 (Full) Fourier series

Fourier series are named in honor of Joseph Fourier (1768-1830), who made important contributions to the study of trigonometric series. Fourier series have many applications such as, solving partial differential equations, signal precessing, image processing.

A function $f(x)$ is **piecewise continuous** in interval (a, b) if we have $a = t_0 < t_1 < ... < t_m = b$, such that f(x) is continuous in each interval (t_i, t_{i+1}) and the limits $\lim_{x\to t_i^-} f(x)$ and $\lim_{x\to t_i^+} f(x)$ exist for all $i = 0, 1, 2, \ldots, m$. In the following, we assume that both f and f' are piecewise continuous.

Definition. Let $f(x)$ be $2L$-periodic function. Then $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \right\}. \tag{13}$$

This series is called the (full) Fourier series for $f(x)$. The coefficients a_n $(n \geq 0)$ are called the Fourier cosine coefficients, and the coefficients b_n $(n \geq 1)$ are called the Fourier sine coefficients.

Remark. The "=" occurs at every $x \in [-L, L]$ where f is continuous. If we omit the condition where f is continuous at x, then we may write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})).$$

Theorem. The Fourier coefficients can be calculated as follows:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L})\, dx \qquad n = 0, 1, 2, \ldots. \tag{14}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L})\, dx \qquad n = 1, 2, 3, \ldots. \tag{15}$$

Proof. The coefficient a_0 is the simplest to find: integrating (23) from $-L$ to L,

$$\begin{aligned} \int_{-L}^{L} f(x)\, dx &= \int_{-L}^{L} \frac{a_0}{2}\, dx + \sum_{n=1}^{\infty} \left\{ a_n \int_{-L}^{L} \cos(\frac{n\pi x}{L})\, dx + a_n \int_{-L}^{L} \sin(\frac{n\pi x}{L})\, dx \right\} \\ &= \int_{-L}^{L} \frac{a_0}{2}\, dx \end{aligned}$$

The series on the right vanishes, and we find that

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx.$$

We do the same thing to compute, say, b_m, except that first we multiply (23) through by $\sin(\frac{m\pi x}{L})$. We get

$$
\begin{aligned}
\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})\,dx &= \int_{-L}^{L} \frac{a_0}{2}\sin(\frac{m\pi x}{L})\,dx + \\
&\sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})\,dx + b_n\int_{-L}^{L}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})\,dx\right\} \\
&= b_m\int_{-L}^{L}\sin(\frac{m\pi x}{L})\sin(\frac{m\pi x}{L})\,dx = b_m L. \Rightarrow
\end{aligned}
$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})\,dx \qquad m = 1, 2, 3, \dots.$$

Likewise we can get the formula for a_m.

Formulas (14) and (15) allow us to compute the Fourier coefficients of f.

Remark 1. Even though f is defined only on $[-L, L]$, the right-hand side of (23) is $2L$-periodic, so we could view f as being defined over the whole line, but $2L$-periodic as well.

Remark 2. If f is even on $[-L, L]$, then $f(x)\sin(\frac{m\pi x}{L})$ is odd on $[-L, L]$, so $b_n = 0$ for all $n \geq 1$; and $f(x)\cos(\frac{m\pi x}{L})$ is even on $[-L, L]$, so

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx \qquad n = 0, 1, 2, \dots.$$

Remark 3. If f is odd on $[-L, L]$, then $f(x)\cos(\frac{m\pi x}{L})$ is odd on $[-L, L]$, so $a_n = 0$ for all $n \geq 0$; and $f(x)\sin(\frac{m\pi x}{L})$ is even on $[-L, L]$, so

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx \qquad n = 1, 1, 2, \dots.$$

3.3 2π-periodic function

If f is 2π-periodic (i.e., $L = \pi$), then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n\sin nx\},$$

where

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. && (16)\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n = 0, 1, 2, \dots. && (17)\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad n = 1, 2, 3, \dots. && (18)
\end{aligned}
$$

Example 45. *Let $f(x) = x$, for all $x \in [-\pi, \pi)$, and $f(x)$ be 2π-periodic. Compute the Fourier coefficients.*

Notice that $\cos(nx)$ is an even function, while f and $\sin(nx)$ are odd functions.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} xdx = 0 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)dx = \frac{2}{\pi}\left(\left[-\frac{x\cos(nx)}{n}\right]_0^{\pi} + \left[\frac{\sin(nx)}{n^2}\right]_0^{\pi}\right) = (-1)^{n+1}\frac{2}{n}
\end{aligned}$$

Notice that a_0, a_n are 0 because x and $x\cos(nx)$ are odd functions. Hence the Fourier series for $f(x) = x$ is:

$$\begin{aligned}
x &= a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)) = \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx), \quad \forall x \in (-\pi, \pi)
\end{aligned}$$

Example 46. *Let $f(x) = x^2$, $x \in [-\pi, \pi)$, and $f(x)$ be 2π-periodic. Compute the Fourier coefficients.*

Since f is *even* ($f(x) = f(-x)$ for all x), then $b_n = 0$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{3}\pi^2,$$

and for $n \geq 1$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx \\
&= \frac{1}{n\pi}\left\{x^2\sin(nx)\big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\sin(nx)\,dx\right\} \\
&= \frac{(-1)^n\cdot 4}{n^2}.
\end{aligned}$$

Thus for $x \in (-\pi, \pi)$,

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n\cdot 4}{n^2}\cos(nx).$$

Example 47. *Let*

$$f(x) = \begin{cases} 0, & \textit{for } x \in [-\pi, 0); \\ 1, & \textit{for } x \in (0, \pi). \end{cases},$$

and let $f(x)$ be 2π-periodic. Find the Fourier series of $f(x)$.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} 1dx\right) = 1, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cos(nx)dx + \int_{0}^{\pi} 1\cos(nx)dx\right) = \frac{1}{n\pi}\sin(nx)|_0^\pi = 0, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\sin(nx)dx + \int_{0}^{\pi} 1\sin(nx)dx\right) = -\frac{1}{n\pi}\cos(nx)|_0^\pi \\ &= \frac{1}{n\pi}\left(1-(-1)^n\right) = \begin{cases} \frac{2}{n\pi}, & \text{for odd } n; \\ 0, & \text{for even } n. \end{cases} \end{aligned}$$

Hence the Fourier series for $f(x)$ is:

$$\begin{aligned} f(x) &\sim \frac{1}{2} + \sum_{odd\ n} \frac{2}{n\pi}\sin(nx) = \\ &= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi}\sin(2n+1)x, \quad \forall x \in (-\pi, \pi). \end{aligned}$$

3.4 Points of discontinuity and convergence

In equation (1), "=" means that the series on the right converges to the function on the left at each point x. It often happens that the Fourier series of a function f fails to converge to that function, in particular at the points of discontinuity of f.

The facts are:

- **The Fourier Theorem**: If the function $f(x)$ is piecewise continuously differentiable then its Fourier series converges for every x to the average value

$$f_{av}(x) = \frac{f(x+) + f(x-)}{2}, \tag{19}$$

  where

$$f(x+) = \lim_{t\to x+} f(t), \quad f(x-) = \lim_{t\to x-} f(t).$$

- At the points where $f(x)$ is continuous, $f_{av}(x) = f(x)$.

All the functions we shall consider in the sequel are piecewise continuously differentiable, and therefore the Fourier series will represent the function. In order to ensure that the Fourier series of function $f(x)$ converges to that function at every $x \in \mathbb{R}$, sometimes it is necessary to redefine $f(x)$ at the points of discontinuity x, so that $f_{av}(x) = f(x)$.

Example 48. *Let*

$$f(x) = \begin{cases} -1, & \text{for } x \in (-\pi, 0); \\ x, & \text{for } x \in (0, \pi); \\ 0, & \text{for } x = 0, \pi, -\pi. \end{cases}$$

Determine the sums to which the series converges at $x = 0, \pm\pi, 88\pi, 101\pi$.

Solution: The sum=-0.5 at 0; $\frac{\pi-1}{2}$ at $\pm\pi$.

3.5 Geometric interpretation of Fourier series

In the example above, if we let

$$S_1 = \frac{1}{2} + \frac{2}{\pi}\sin x,$$

$$S_3 = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x,$$

$$S_5 = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x,$$

each partial sum is a continuous function that approximates the discontinuous function f(x) on the interval $(-\pi, \pi)$. The bigger n, the better the approximation.

3.6 General $2L$-periodic function

Example 49. *Let* $f(x)$ *be 2-periodic and* $f(x) = \begin{cases} x, & 0 < x < 1; \\ 0, & -1 < x \leq 0. \\ 0.5, & x = -1, 1. \end{cases}$

In this case, $L = 1$.

$$\begin{aligned} a_0 &= \int_0^1 x\,dx = \frac{1}{2}, \\ a_n &= \int_0^1 x\cos(n\pi x)\,dx = \frac{(-1)^n - 1}{(n\pi)^2}, \\ b_n &= \int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^n}{n\pi}. \end{aligned}$$

The full Fourier series is

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{(n\pi)^2}\cos(n\pi x) + \frac{(-1)^n}{n\pi}\sin(n\pi x)\right].$$

3.7 Fourier cosine and sine Series on interval $(0, L)$: Even and odd extensions

Half-range Expansions

Let $f(x)$ be define on $(0, L)$. Three special extensions are important:
(i) Consider $f(x)$ as an odd function on $(-L, L)$, i.e.,

$$f_{odd}(x) = \begin{cases} f(x), & x \in (0, L); \\ -f(-x), & x \in (-L, 0). \end{cases}$$

Then $f(x)$ is $2L-$periodic. Thus $a_n = 0$ for all n, (23) becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{20}$$

which is called **Fourier sine series** of f, where

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right) dx \qquad m = 1, 2, 3, \ldots.$$

(ii) Consider $f(x)$ as an even function on $(-L, L)$, i.e.,

$$f_{even}(x) = \begin{cases} f(x), & x \in (0, L); \\ f(-x), & x \in (-L, 0). \end{cases}$$

Then $f(x)$ is $2L-$periodic. Thus $b_n = 0$ for all n, (23) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{21}$$

which is called **Fourier cosine series** of f, where

$$a_m = \frac{2}{L}\int_0^L f(x)\cos(\frac{m\pi x}{L})\, dx \qquad m = 0, 1, 2, 3, \ldots.$$

The cosine and sine series here are known as HALF-RANGE EXPANSIONs.

(iii) Consider $f(x)$ as $L-$periodic. Then half-period is $L/2$.

$$a_n = \frac{1}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L/2})\, dx \qquad n = 0, 1, 2, \ldots.$$

$$b_n = \frac{1}{L/2}\int_0^L f(x)\sin(\frac{n\pi x}{L/2})\, dx \qquad n = 1, 2, 3, \ldots.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)] \tag{22}$$

which is called **(full) Fourier series** of f.

Example 50. *Let* $f(x) = \begin{cases} x, & 0 \le x < 1. \\ 0, & 1 \le x < 2; \end{cases}$ *. Find the Fourier sine series and Fourier cosine series.*

Solution: (i) Fourier sine series: for $m = 1, 2, 3, \ldots,$

$$\begin{aligned} b_m &= \frac{2}{L}\int_0^L f(x)\sin(\frac{m\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\sin(\frac{m\pi x}{2})\,dx \\ &= \int_0^1 x\sin(\frac{m\pi x}{2})\,dx = \left[-\frac{2}{m\pi}x\cos(\frac{m\pi x}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi x}{2})\right]_0^1 \\ &= -\frac{2}{m\pi}\cos(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi}{2}). \end{aligned}$$

(ii) Fourier cosine series:for $m = 0, 1, 2, 3, \ldots,$

$$\begin{aligned} a_m &= \frac{2}{L}\int_0^L f(x)\cos(\frac{m\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\cos(\frac{m\pi x}{2})\,dx \\ &= \int_0^1 x\cos(\frac{m\pi x}{2})\,dx = \left[\frac{2}{m\pi}x\sin(\frac{m\pi x}{2}) + \frac{4}{m^2\pi^2}\cos(\frac{m\pi x}{2})\right]_0^1 \\ &= \frac{2}{m\pi}\sin(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\cos(\frac{m\pi}{2}) - \frac{4}{m^2\pi^2}. \end{aligned}$$

Remark. (i) 2-periodic extension of f:

$$\tilde{f}(x) = \begin{cases} x, & 0 \le x < 1; \\ 0, & 1 \le x < 2. \\ \tilde{f}(x+2) = \tilde{f}(x), & \text{for any } x. \end{cases}$$

In this case, $L = 1$.

$$\begin{aligned} a_0 &= \int_0^1 x\,dx = \frac{1}{2}, \\ a_m &= \int_0^1 x\cos(m\pi x)\,dx = \frac{(-1)^m - 1}{(m\pi)^2}, \\ b_m &= \int_0^1 x\sin(m\pi x)\,dx = \frac{(-1)^m}{m\pi}. \end{aligned}$$

The full Fourier series is

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{(n\pi)^2}\cos(n\pi x) + \frac{(-1)^n}{n\pi}\sin(n\pi x)\right].$$

At $x = 2$, this series converges to

$$\frac{\tilde{f}(2+) + \tilde{f}(2-)}{2} = \tilde{f}(2) = 0;$$

At $x = 1$, this series converges to

$$\frac{\tilde{f}(1+) + \tilde{f}(1-)}{2} = \frac{1+0}{2} = 0.5$$

At $x = -1$, this series converges to

$$\frac{\tilde{f}(-1+) + \tilde{f}(-1-)}{2} = \frac{1+0}{2} = 0.5.$$

(ii) 4-periodic **odd** extension of f:

$$\tilde{f}(x) = \begin{cases} 0, & -2 \le x < -1; \\ x, & -1 \le x < 1; \\ 0, & 1 \le x < 2. \\ \tilde{f}(x+4) = \tilde{f}(x), & \text{otherwise.} \end{cases}$$

In this case, $L = 2$.

$$\begin{aligned} a_0 &= 0, \\ a_m &= 0, \\ b_m &= \frac{2}{2}\int_0^2 f(x)\sin(\frac{m\pi x}{2})\,dx = -\frac{2}{m\pi}\cos(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi}{2}). \end{aligned}$$

The full Fourier series is

$$f(x) = \sum_{m=1}^{\infty}\left[-\frac{2}{m\pi}\cos(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi}{2})\right]\sin(\frac{m\pi x}{2}),$$

which is the Fourier sine series. At $x = 2$, this series converges to

$$\frac{\tilde{f}(2+) + \tilde{f}(2-)}{2} = \tilde{f}(2) = 0;$$

At $x = 1$, this series converges to

$$\frac{\tilde{f}(1+) + \tilde{f}(1-)}{2} = \frac{1+0}{2} = 0.5$$

At $x = -1$, this series converges to

$$\frac{\tilde{f}(-1+) + \tilde{f}(-1-)}{2} = \frac{0-1}{2} = -0.5.$$

(iii) 4-periodic **even** extension of f:

$$\tilde{f}(x) = \begin{cases} 0, & -2 \le x < -1; \\ -x, & -1 \le x < 0; \\ x, & 0 \le x < 1; \\ 0, & 1 \le x < 2. \\ \tilde{f}(x+4) = \tilde{f}(x), & \text{otherwise.} \end{cases}$$

In this case, $L = 2$.

$$\begin{aligned} a_m &= = \frac{2}{2}\int_0^2 f(x)\cos(\frac{m\pi x}{2})\,dx = \frac{2}{m\pi}\sin(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\cos(\frac{m\pi}{2}) - \frac{4}{m^2\pi^2}, \\ b_m &= 0 \end{aligned}$$

The full Fourier series is

$$f(x) = \sum_{m=1}^{\infty}\left[\frac{2}{m\pi}\sin(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\cos(\frac{m\pi}{2}) - \frac{4}{m^2\pi^2}\right]\cos(\frac{m\pi x}{2}),$$

which is the Fourier cosine series. At $x = 2$, this series converges to

$$\frac{\tilde{f}(2+) + \tilde{f}(2-)}{2} = \tilde{f}(2) = 0;$$

At $x = 1$, this series converges to

$$\frac{\tilde{f}(1+) + \tilde{f}(1-)}{2} = \frac{1+0}{2} = 0.5$$

At $x = -1$, this series converges to

$$\frac{\tilde{f}(-1+) + \tilde{f}(-1-)}{2} = \frac{0+1}{2} = 0.5.$$

3.8 (Full) Fourier Series on Interval $[a, b]$

Let f be defined on $[a, b]$ and $f(a) = f(b)$. Then we can extend f to a periodic function by means of periodicity condition $f(x + 2(b - a)) = f(x)$ for all $x \in \mathbb{R}$. If we consider $2L = b - a$, then $L = \frac{b-a}{2}$. The full Fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})) \tag{23}$$

at every $x \in [a, b]$ where f is continuous. The coefficients a_n ($n \ge 0$) and the coefficients b_n ($n \ge 1$) are calculated as follows:

$$a_n = \frac{1}{L}\int_a^b f(x)\cos(\frac{n\pi x}{L})\,dx \qquad n = 0, 1, 2, \ldots.$$

$$b_n = \frac{1}{L}\int_a^b f(x)\sin(\frac{n\pi x}{L})\,dx \qquad n = 1, 2, 3, \ldots.$$

Example 51. *Let* $f(x) = \begin{cases} -1, & -\pi < x < 0; \\ x, & 0 < x < \pi; \\ 0, & x = 0, -\pi, \pi; \end{cases}$

Then

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -1dx + \int_0^{\pi} xdx\right) = \frac{\pi-2}{2}, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos(nx)dx + \int_0^{\pi} x\cos(nx)dx\right) \\
&= -\frac{1}{n\pi}\sin(nx)|_{-\pi}^{0} + \left(\frac{1}{n\pi}x\sin(nx)|_0^{\pi} + \frac{1}{n^2\pi}\cos(nx)|_0^{\pi}\right) = \frac{1}{n^2\pi}[(-1)^n - 1] \\
&= \begin{cases} -\frac{2}{n^2\pi}, & \text{for odd } n; \\ 0, & \text{for even } n. \end{cases}, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(nx)dx + \int_0^{\pi} x\sin(nx)dx\right) \\
&= \frac{1}{n\pi}\cos(nx)|_{-\pi}^{0} + \left(-\frac{1}{n\pi}x\cos(nx)|_0^{\pi} + \frac{1}{n^2\pi}\sin(nx)|_0^{\pi}\right) \\
&= \begin{cases} \frac{2+\pi}{n\pi}, & \text{for odd } n; \\ -\frac{1}{n}, & \text{for even } n. \end{cases}
\end{aligned}$$

Note that

$$f_{av}(\pm 2n\pi) = -0.5, \quad f_{av}(\pm(2n-1)\pi) = \frac{\pi - 1}{2}.$$

To ensure that the Fourier series converges to f(x) at every point x, we need to redefine f as: $f(x) = \begin{cases} -1, & -\pi \le x < 0; \\ x, & 0 \le x < \pi; \\ -0.5, & x = 0; \\ \frac{\pi-1}{2}, & x = -\pi, \pi; \end{cases}$

3.9 *Periodic Solution of non-homogeneous equations

Theorem. Consider the equation

$$y'' + \lambda y = f(x), \quad -\infty < x < \infty,$$

where $\lambda \in \mathbb{R}$, f is $2L$-periodic and continuous, f' and f'' are piecewise continuous. Then we can find $2L$-periodic solution. Let

$$y = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n\cos(\frac{n\pi x}{L}) + d_n\sin(\frac{n\pi x}{L})). \tag{24}$$

Then

$$\lambda c_0 = a_0, \quad \left(\lambda - \frac{n^2\pi^2}{L^2}\right) c_n = a_n, \quad \left(\lambda - \frac{n^2\pi^2}{L^2}\right) d_n = b_n. \tag{25}$$

Remark.

- If $\lambda = 0$, then there is no periodic solution, unless $a_0 = 0$, in which case, c_0 is arbitrary.
- If $\lambda = \frac{n^2\pi^2}{L^2}$, then there is no periodic solution, unless $a_n = b_n = 0$, in which case c_n and d_n are arbitrary.

Example 52. *Let* $f(x) = \begin{cases} x - x^2, & 0 \le x \le 1; \\ odd, & \\ 2-periodic. & \end{cases}$ *Find a 2-periodic solution of the equation*

$$y'' + y = f(x), \quad -\infty < x < \infty.$$

Solution: Here $L = 1$, $\lambda = 1$. Since f is odd, $a_n = 0$ for $n \ge 0$.

$$b_n = 2\int_0^1 (x - x^2)\sin(n\pi x)dx = \frac{4[1-(-1)^n]}{n^3\pi^3}, \quad n \ge 1.$$

This implies that $c_n = 0$ for all n and

$$d_n = \frac{4[1-(-1)^n]}{n^3\pi^3(1-n^2\pi^2)}.$$

$$y = \sum_{n=1}^{\infty} \frac{4[1-(-1)^n]}{n^3\pi^3(1-n^2\pi^2)} \sin(n\pi x).$$

Theorem (Boundary-value problems. Consider the equation

$$y'' + \lambda y = f(x), \quad 0 < x < M,$$

where $\lambda \in \mathbb{R}$, f is continuous, f' and f'' are piecewise continuous on [0,M].

(1) If $y(0) = y(M) = 0$, then it has a Fourier sine series solution with $L = M$;

(2) If $y'(0) = y'(M) = 0$, then it has a Fourier cosine series solution with $L = M$;

(3) If $y(0) = y(M/2)$ and $y'(0) = y'(M)$, then it has a full Fourier series solution with $L = M/2$.

Example 6: Solve BVP:

$$y'' + y = x, \quad 0 < x < 1, y'(0) = y'(1) = 0.$$

Solution: This is Case (2).

$$b_n = 0, a_0 = 1, \quad a_n = \frac{2[(-1)^n - 1]}{n^2\pi^2}, \quad n \ge 1. \Rightarrow$$

$$d_n = 0, c_0 = 1, \quad c_n = \frac{2[(-1)^n - 1]}{n^2\pi^2(1-n^2\pi^2)}, \quad n \ge 1.$$

4 IV Partial Differential Equations

4.1 What is a Partial Differential Equation?

A partial differential equation (PDE) relates the partial derivatives of a function of two or more independent variables together. For example, Laplace's equation for $\Phi(x, y)$,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{26}$$

arises in many places in mathematics and physics. For simplicity, we will use subscript notation for partial derivatives, so this equation can also be written $\Phi_{xx} + \Phi_{yy} = 0$.

We say a function is a **solution** to a PDE if it satisfy the equation and any side conditions given. Mathematicians are often interested in if a solution **exists** and when it is **unique**.

Example. $\Phi_1 = x$ and $\Phi_2 = x^2 - y^2$ are solutions to Laplace's equation (26).

The **order** of PDE is the highest partial derivative that appears in the equation. So, Laplace's Equation (26) is second-order.

We also define **linear** PDE's as equations for which the dependent variable (and its derivatives) appear in terms with degree at most one. Anything else is called **nonlinear**.

Linear equations can further be classified as **homogeneous** for which the dependent variable (and it derivatives) appear in terms with degree **exactly** one, and non-homogeneous which may contain terms which only depend on the independent variable.

If two solutions, say u_1 and u_2, satisfy a linear homogeneous PDE, that any linear combination of them

$$u = c_1 u_1 + c_2 u_2$$

is also a solution. So, for example, since

$$\Phi_1 = x^2 - y^2 \qquad \Phi_2 = x$$

both satisfy Laplace's equation, $\Phi_{xx} + \Phi_{yy} = 0$, so does any linear combination of them

$$\Phi = c_1\Phi_1 + c_2\Phi_2 = c_1(x^2 - y^2) + c_2 x.$$

This property is extremely useful for constructing solutions which satisfy certain initial conditions and boundary conditions.

4.2 Heat Equation or Diffusion Equation

When we derived Newton's Law of cooling we made several assumptions – most importantly that the temperature does not vary with location. If we account for the variation of temperature with location, we can derive a PDE called the **heat equation** or, more generally, the **diffusion equation**. Suppose we consider a metal bar, with a uniform cross-sectional area, A, whose temperature, $u(x,t)$, is a function of time, t, and the position, x, along the bar (that is we assume the temperature is uniform in every cross-section). Then $u(x,t)$ satisfies the diffusion equation (DE),

$$u_t = c^2 u_{xx}, \tag{27}$$

where c^2 is a constant known as the **thermal diffusivity**, determined by the geometry and physical properties of the metal bar.

Suppose we consider a bar of finite length L, occupying the region $0 < x < L$. At the boundaries of the metal bar we can specify a fixed temperature,

$$u(0,t) = u_0 \qquad u(L,t) = u_1, \tag{28}$$

which are usually referred to as Dirichlet **boundary conditions (BCs)**.

We also need to specify the initial temperature distribution,

$$\textbf{Initial condition (IC)}: \quad u(x,0) = f(x) \qquad 0 \le x \le L. \tag{29}$$

4.2.1 Homogeneous Boundary Conditions: Separation of Variables

Theorem 11. *The solution of the heat equation*

$$\begin{aligned} u_t = c^2 u_{xx} \qquad & 0 < x < L, t > 0 \qquad PDE \\ u(0,t) = 0 \quad u(L,t) = 0 \qquad & t > 0 \qquad BC \\ u(x,0) = f(x) \qquad & 0 < x < L, \qquad IC \end{aligned}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t},$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{30}$$

Proof. The method to solve the wave equation comprises three steps.

Step 1: Separation of variables.

Let $u(x,t) = X(x)T(t)$ and substitute it into the equation we have

$$XT_t = c^2 X_{xx} T$$

and dividing by XT we find

$$\frac{T_t}{c^2 T} = \frac{X_{xx}}{X} = \lambda. \tag{31}$$

where λ is to be determined. Now because T_t/DT is **only** a function of t and X_{xx}/X is **only** a function of x we know that λ must be independent of x and t respectively, and therefore must be a constant – consequently it is known as the **separation constant**. We can now solve the resulting ODE for $T(t)$

$$T_t = \lambda c^2 T \quad \Rightarrow \quad T(t) = Ce^{\lambda D t}. \tag{32}$$

Step 2: Eigenfunction.

We now look for a solution for the $X(x)$ equation that also satisfies the homogeneous boundary conditions. From the boundary conditions (BC), we know that

$$u(0,t) = X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0 \tag{33}$$

$$u(L,t) = X(L)T(t) = 0 \quad \Rightarrow \quad X(L) = 0 \tag{34}$$

So finally we conclude that we are looking for solutions to the **Boundary Value Problem** for $X(x)$,

$$X_{xx} - \lambda X = 0, \qquad X(0) = X(L) = 0. \tag{35}$$

Now we consider three cases:

1. If $\lambda = 0$, then $X(x) = Ax+B$, it follows from $X(0) = 0$ that $B = 0$. Thus $X(x) = Ax$. From $X(L) = 0$ we have $A = 0$, so $X(x) = 0$, trivial solution.

2. If $\lambda > 0$, then $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$, it follows from $X(0) = X(L) = 0$ that $A + B = 0$, $Ae^{\sqrt{\lambda}L} + Be^{-\sqrt{\lambda}L} = 0$. Thus $A = B = 0$ and so $X(x) \equiv 0$, a trivial solution.

3. If $\lambda < 0$, then $X(x) = A\cos(\sqrt{-\lambda}x) + B\sin(\sqrt{-\lambda}x)$. Applying the boundary conditions we see that $X(0) = 0$ implies that $A = 0$, and that $B\sin(\sqrt{-\lambda}L) = 0$. To have non-trivial solution, $B \neq 0$, so $\sin(\sqrt{-\lambda}L) = 0, \Rightarrow \sqrt{-\lambda} = \frac{n\pi}{L} \Rightarrow$

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, ...$$

where B_n are constants. Note that $\lambda = -(\frac{n\pi}{L})^2$. These special values of λ are called **eigenvalues** and the associated functions, $X_n(x)$, are known as **eigenfunctions**. This implies

$$T_n(t) = C_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}.$$

Multiplying the solution for $X_n(x)$ and $T_n(t)$ together finally yields a solution for $u_n(x,t)$ satisfying BCs $u_n(0,t) = u_n(L,t) = 0$ is

$$u_n(x,t) \equiv b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t} \quad n = 1, 2, 3 \ldots. \tag{36}$$

Step 3: Superposition. We use superposition of the eigenfunctions to satisfy the initial condition.

To consider the IC: $u(x,0) = f(x)$, we let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}, \Rightarrow$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \Rightarrow$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

♠

Example 53. *Solve*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t) = u(1,t) = 0,$$

and the initial condition

$$u(x,0) = f(x) = \begin{cases} x & \text{if } 0 < x < 1/2, \\ 1 - x & \text{if } 1/2 < x < 1. \end{cases}$$

Solution: $c = 1$, $L = 1$. Thus

$$b_n = 2\left(\int_0^{1/2} x \sin(n\pi x) dx + \int_{1/2}^1 (1-x) \sin(n\pi x) dx\right). \tag{37}$$

Integration by parts yields,

$$b_{2m} = (-1)^m \frac{1}{m^2\pi^2}, \quad b_{2m+1} = 0,$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t} = \sum_{m=1}^{\infty} (-1)^m \frac{1}{m^2\pi^2} \sin(2m\pi x) e^{-(2m\pi)^2 t}$$

4.2.2 Non-homogeneous Boundary Conditions

THE DIRICHLET PROBLEM FOR THE DIFFUSION EQUATION
(NON-HOMOGENEOUS BOUNDARY CONDITIONS)

$$\begin{array}{rlll} & u_t = \alpha^2 u_{xx} & 0 < x < L, t > 0 & PDE \\ u(0,t) = A & u(L,t) = B & t > 0 & BC \\ & u(x,0) = f(x) & 0 \le x \le L. & IC \end{array}$$

By letting

$$u(x,t) = v(x) + w(x,t),$$

we see that $w(x,t)$ satisfies Homogeneous Boundary Conditions and $v(x)$ satisfies

$$v''(x) = 0, \quad v(0) = A, v(L) = B.$$

Theorem 12. *The solution of the heat equation (27) subject to BCs: $u(0,t) = A$, $u(L,t) = B$ and IC: $u(x,0) = f(x)$ is given by*

$$u(x,t) = v(x) + w(x,t)$$

with

$$v(x) = \frac{B-A}{L}x + A, \quad w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t},$$

where b_n satisfies

$$b_n = \frac{2}{L}\int_0^L [f(x) - v(x)] \sin\left(\frac{n\pi x}{L}\right) dx. \tag{38}$$

Example. Find the solution of the Heat Equation $u_{xx} = 9u_t$, subject to BCs: $u(0,t) = 1$, $u(2,t) = 3$ and IC: $u(x,0) = f(x)$, where

$$f(x) = \begin{cases} 2x+1, & 0 \le x < 1, \\ x+1, & 1 \le x < 2, \end{cases}.$$

Solution: Here $\alpha = \frac{1}{3}$, $L = 2$, $A = 1$, $B = 3$. Thus $v(x) = x + 1$. By (38), for $m = 1, 2, 3, \ldots,$

$$\begin{aligned} b_m &= \frac{2}{L}\int_0^L [f(x) - v(x)] \sin(\frac{m\pi x}{L})\, dx = \frac{2}{2}\int_0^2 [f(x) - v(x)] \sin(\frac{m\pi x}{2})\, dx \\ &= \int_0^1 x \sin(\frac{m\pi x}{2})\, dx = \left[-\frac{2}{m\pi} x \cos(\frac{m\pi x}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi x}{2})\right]_0^1 \\ &= -\frac{2}{m\pi}\cos(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi}{2}). \end{aligned}$$

$$u(x,t) = x + 1 + \sum_{n=1}^{\infty} \left[-\frac{2}{n\pi} \cos(\frac{n\pi}{2}) + \frac{4}{n^2\pi^2} \sin(\frac{n\pi}{2}) \right] \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi}{6}\right)^2 t}.$$

4.2.3 *The bar with insulated ends

$$\begin{aligned} u_t &= \alpha^2 u_{xx} & 0 < x < L, t > 0 & \qquad PDE \\ u_x(0,t) = u_x(L,t) &= 0 & t > 0 & \qquad BC \\ u(x,0) &= f(x) & 0 \le x \le L. & \qquad IC \end{aligned}$$

The solution of this PDE is:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}, \tag{39}$$

where a_n satisfies

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx. \tag{40}$$

4.3 The Wave Equation

The wave equation has the form:

$$u_{tt} = c^2 u_{xx}, \tag{41}$$

where $u = u(x,t)$ can be thought of as the vertical displacement of the vibration of a string.

4.3.1 Dirichlet Problem and Separation of Variables

If we tie the string at both ends we can have the following boundary conditions:

$$u(0,t) = A(t), u(L,t) = B(t),$$

where A, B are piecewise functions. For example, we can have a sinusoidal function at one end and a Heaviside function at the other.

When the boundary values A and B are 0 we obtain the **Dirichlet Problem** for the wave equation:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & 0 < x < L, t > 0 & \qquad PDE \\ u(0,t) = 0, u(L,t) &= 0, & t > 0 & \qquad BC \\ u(x,0) = f(x), u_t(x,0) &= g(x) & 0 < x < L & \qquad IC. \end{aligned}$$

As you have seen in Heat equation, the method of separating the variables is a very convenient way to obtain solutions for PDEs. In the case of the Dirichlet Problem we will quickly review the method.

Theorem 13. *The solution of the above Dirichlet Problem (wave problem) is given by:*

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L}) \right] \sin(\frac{n\pi x}{L}),$$

where:

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Proof. The method to solve the wave equation comprises three steps.

Step 1: Separation of variables.

Let $u(x,t) = X(x)T(t)$ and substitute it into the equation $u_{tt} = c^2 u_{xx}$, to obtain:

$$X(x)T''(t) = c^2 X''(x)T(t),$$

or $\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}$, thus the equality is one of functions of different variables, so both quotients have to be constant.

Say

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = \pm\lambda^2,$$

then we can solve each ordinary differential equation separately. We have the following three cases: $-\lambda^2$, λ^2, and $\lambda = 0$.

Step 2: Eigenfunctions.

Case 1 When the constant is λ^2, then the solutions for $\frac{X''(x)}{X(x)} = \lambda^2$, are: $X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$, and the solutions for $\frac{T''(t)}{c^2T(t)} = \lambda^2$, are $T(t) = d_1 e^{\lambda ct} + d_2 e^{-\lambda ct}$. Then

$$u(x,t) = (d_1 e^{\lambda ct} + d_2 e^{-\lambda ct})(c_1 e^{\lambda x} + c_2 e^{-\lambda x}).$$

Let's take a look at the boundary conditions: $u(0,t) = 0, u(L,t) = 0$. We imply that $c_1 = c_2 = 0$, and so $u(x,t) = 0$, only trivial solution.

Case 2 When the constant is 0, then the equations become $X''(x) = T''(t) = 0$, and $X(x) = c_1 x + c_2$, and $T(t) = d_1 t + d_2$. Then

$$u(x,t) = (d_1 t + d_2)(c_1 x + c_2).$$

Let's take a look at the boundary conditions: $u(0,t) = 0, u(L,t) = 0$. We imply that $c_1 = c_2 = 0$, and so $u(x,t) = 0$, only trivial solution.

Case 3 When the constant is $-\lambda^2$, then the solutions for $\frac{X''(x)}{X(x)} = -\lambda^2$, are:

$$X(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x),$$

and the solutions for $\frac{T''(t)}{c^2 T(t)} = -\lambda^2$, are:

$$T(t) = d_1 \sin(\lambda ct) + d_2 \cos(\lambda ct).$$

Then

$$u(x,t) = (d_1 \sin(\lambda ct) + d_2 \cos(\lambda ct))(c_1 \sin(\lambda x) + c_2 \cos(\lambda x)).$$

The boundary conditions $u(0,t) = u(L,t) = 0$ translate into:

$$\begin{aligned} (d_1 \sin(\lambda ct) + d_2 \cos(\lambda ct))(c_1 \sin(0) + c_2 \cos(0)) &= 0 \\ (d_1 \sin(\lambda ct) + d_2 \cos(\lambda ct))(c_1 \sin(\lambda L) + c_2 \cos(\lambda L)) &= 0, \quad \forall t > 0, \end{aligned}$$

namely:

$$\begin{aligned} c_2 &= 0 \\ c_1 \sin(\lambda L) &= 0. \end{aligned}$$

From the last condition we obtain $\lambda = \frac{\pi n}{L}$, and

$$u_n(x,t) = \left[d_{1n} \sin(\frac{\pi n}{L} ct) + d_{2n} \cos(\frac{\pi n}{L} ct) \right] c_n \sin(\frac{\pi n}{L} x).$$

Step 3: Superposition. We use superposition of the eigenfunctions to satisfy the initial condition.

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) &= \sum_{n=1}^{\infty} \left[d_{1n} \sin(\frac{\pi nct}{L}) + d_{2n} \cos(\frac{\pi nct}{L}) \right] c_n \sin(\frac{\pi nx}{L}) \\ &= \sum_{n=1}^{\infty} \left[a_n \cos(\frac{\pi nct}{L}) + b_n \sin(\frac{\pi nct}{L}) \right] \sin(\frac{\pi nx}{L}). \end{aligned}$$

The only conditions left to check are the initial conditions:

$$\begin{aligned} u(x,0) &= f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \\ u_t(x,0) &= g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

From the 1st one we have

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

From the 2nd one we have

$$b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Then

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L})\right]\sin(\frac{n\pi x}{L}).$$

Example 54. *Solve the following equation:*

$$\begin{aligned} 9u_{tt} = u_{xx}, \qquad & 0 < x < 4, t > 0 \qquad && DE \\ u(0,t) = 0, u(4,t) = 0, \qquad & t > 0 \qquad && BC \\ u(x,0) = f(x), u_t(x,0) = 0 \qquad & 0 < x < 4 \qquad && IC, \end{aligned}$$

where

$$f(x) = \begin{cases} 2x, & 0 \le x \le 2 \\ -2x+4, & 2 < x \le 4 \end{cases}$$

Solution: $b_n = 0$. Note that

$$f'(x) = \begin{cases} 2, & 0 \le x \le 2 \\ -2, & 2 \le x \le 4, \end{cases}$$

and $f(0) = f(4) = 0$, thus

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)sin(\frac{n\pi x}{L})dx \\ &= \frac{2}{L}f(x)(\frac{-L}{n\pi})\cos(\frac{n\pi x}{L})|_0^L + \frac{2}{L}\frac{L}{n\pi}\int_0^L f'(x)\cos(\frac{n\pi x}{L})dx \\ &= \frac{2}{4}(\frac{4}{n\pi})\int_0^2 2\cos(\frac{n\pi x}{4})dx + \frac{2}{4}(\frac{4}{n\pi})\int_2^4 (-2)\cos(\frac{n\pi x}{4})dx \\ &= \frac{32}{\pi^2 n^2}\sin(\frac{n\pi}{2}). \end{aligned}$$

Now we can write the formal solution to the plucked string equation:

$$u(x,t) = \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(\frac{n\pi}{2})\cos(\frac{n\pi t}{12})\sin(\frac{n\pi x}{4}).$$

***String with free end:**

Example 55. *Solve the following equation:*

$$\begin{aligned} u_{tt} &= 9u_{xx}, & 0 < x < 2, t > 0 && DE \\ u(0,t) = 0, u_x(2,t) &= 0, & t > 0 && BC \\ u(x,0) = 2\sin(\frac{5\pi x}{4}), u_t(x,0) &= 0 & 0 < x < 2 && IC. \end{aligned}$$

Solution: Let $u(x,t) = X(x)T(t)$ and substitute it into the equation $u_{tt} = 9u_{xx}$, to obtain:

$$\frac{T''(t)}{9T(t)} = \frac{X''(x)}{X(x)} = \pm\lambda^2.$$

Case 1 When the constant is λ^2, then the solutions for $\frac{X''(x)}{X(x)} = \lambda^2$, are: $X(x) = c_1e^{\lambda x} + c_2e^{-\lambda x}$. From $u(0,t) = 0, u_x(2,t) = 0$ we have $X(0) = 0, X'(2) = 0$, which gives

$$c_1 + c_2 = 0, c_1\lambda x - c_2\lambda = 0, \Rightarrow c_1 = c_2 = 0, \Rightarrow X(x) = 0, \Rightarrow u(x,t) = 0.$$

So only trivial solution.

Case 2 When the constant is 0, then the equations become $X''(x) = T''(t) = 0$, and so $X(x) = c_1x + c_2$. From $X(0) = 0, X'(2) = 0$, we imply that

$$c_1 = c_2 = 0, \Rightarrow X(x) = 0, \Rightarrow u(x,t) = 0.$$

So only trivial solution.

Case 3 When the constant is $-\lambda^2$, then the solutions for $\frac{X''(x)}{X(x)} = -\lambda^2$, are:

$$X(x) = c_1\sin(\lambda x) + c_2\cos(\lambda x),$$

and the solutions for $\frac{T''(t)}{9T(t)} = -\lambda^2$, are:

$$T(t) = d_1\sin(3\lambda t) + d_2\cos(3\lambda t).$$

Then

$$u(x,t) = (d_1\sin(3\lambda t) + d_2\cos(3\lambda t))(c_1\sin(\lambda x) + c_2\cos(\lambda x)).$$

$$X(0) = 0, \Rightarrow c_2 = 0, \Rightarrow X(x) = c_1\sin(\lambda x).$$

$$X'(2) = 0 \Rightarrow c_1\lambda\cos(2\lambda) = 0 \Rightarrow \lambda = \frac{(2n+1)\pi}{4} \Rightarrow$$

$$X(x) = c_1\sin(\frac{(2n+1)\pi}{4}x), \quad T(t) = d_1\sin(\frac{3(2n+1)\pi}{4}t) + d_2\cos(\frac{3(2n+1)\pi}{4}t) \Rightarrow$$

$$u_n(x,t) = [A_n\sin(\frac{3(2n+1)\pi}{4}t) + B_n\cos(\frac{3(2n+1)\pi}{4}t)]\sin(\frac{(2n+1)\pi}{4}x), \Rightarrow$$

$$u(x,t)=\sum_{n=1}^{\infty}\left[A_n\sin(\frac{3(2n+1)\pi}{4}t)+B_n\cos(\frac{3(2n+1)\pi}{4}t)\right]\sin(\frac{(2n+1)\pi}{4}x).$$

The only conditions left to check are the initial conditions:

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin(\frac{(2n+1)\pi}{4}x)=2\sin(\frac{5\pi x}{4}),\Rightarrow$$

$$B_2=2,B_n=0(n\neq 2),\Rightarrow$$

$$u(x,t)=\left[\sum_{n=1}^{\infty}A_n\sin(\frac{3(2n+1)\pi}{4}t)+2\cos(\frac{15\pi}{4}t)\right]\sin(\frac{(2n+1)\pi}{4}x).$$

$$u_t(x,0)=\sum_{n=1}^{\infty}A_n\frac{3(2n+1)\pi}{4}\sin(\frac{(2n+1)\pi}{4}x)=0,\Rightarrow A_n=0,\Rightarrow$$

$$u(x,t)=2\cos(\frac{15\pi}{4}t)\sin(\frac{(2n+1)\pi}{4}x).$$

4.4 Laplace Equation

Now we consider the following **Dirichlet Problem** for the Laplace equation:

$$\begin{aligned} PDE:&\quad \nabla^2u=\tfrac{\partial^2u}{\partial x^2}+\tfrac{\partial^2u}{\partial y^2}= \quad u_{xx}+u_{yy}=0,\quad (x,y)\in R;\\ BC:&\quad u(x,y)=f(x,y),\quad (x,y)\in\partial R.\end{aligned}$$

R is a region. We can consider $u(x,y)$ as the steady-state (time-independent) temperature distribution.

4.4.1 Solve the Laplace's equation inside a rectangle

Let

$$R=\{(x,y):0<x<L,0<y<M\}.$$

Case 1: Consider the boundary-value problem

$$\begin{aligned} PDE:&\quad u_{xx}+u_{yy}=0,\quad (x,y)\in R;\\ BC:&\quad u(x,0)=u(x,M)=u(0,y)=0,u(L,y)=f(y).\end{aligned}$$

We will use the method of separation of variables again.

Let $u(x,y)=X(x)Y(y)$ and substitute in the equation to obtain:

$$\frac{X''}{X}=-\frac{Y''}{Y}=constant=\lambda^2$$

(as before, when constant is 0 or negative, no non-trivial solution!).

$$\Rightarrow \begin{cases} X'' - \lambda^2 X = 0 \\ Y'' + \lambda^2 Y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \\ Y = c_3 \sin \lambda y + c_4 \cos \lambda y \end{cases}$$

$$\Rightarrow u(x,y) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 \sin \lambda y + c_4 \cos \lambda y).$$

The boundary conditions

$$u(x,0) = 0 \Rightarrow c_4 = 0; \quad u(x,M) = 0 \Rightarrow c_3 \sin \lambda M = 0;$$

$$u(0,y) = 0, \Rightarrow c_2 = -c_1.$$

$$\Rightarrow u(x,y) = c_1(e^{\lambda x} - e^{-\lambda x}) c_3 \sin \lambda y = 2c_1 c_3 \sinh(\lambda x) \sin \lambda y.$$

To have non-trivial solution, $\sin \lambda M = 0, \Rightarrow \lambda = \frac{n\pi}{M}$, where n is integer.

$$\Rightarrow u_n(x,y) = a_n \sinh\left(\frac{n\pi x}{M}\right) \sin\left(\frac{n\pi y}{M}\right) \Rightarrow .$$

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi x}{M}\right) \sin\left(\frac{n\pi y}{M}\right). \tag{42}$$

By the boundary condition

$$f(y) = u(L,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi L}{M}\right) \sin\left(\frac{n\pi y}{M}\right)$$

we imply that

$$a_n = \frac{2}{M \sinh\left(\frac{n\pi L}{M}\right)} \int_0^M f(y) \sin\left(\frac{n\pi y}{M}\right) dy. \tag{43}$$

Case 2: Consider the boundary-value problem

$$\begin{aligned} PDE: &\quad u_{xx} + u_{yy} = 0, \quad (x,y) \in R; \\ BC: &\quad u(x,0) = u(x,M) = u(L,y) = 0, u(0,y) = f(y). \end{aligned}$$

Let $z = L - x, \quad u(x,y) = w(z,y)$. Then $x = 0 \leftrightarrow z = L; x = L \leftrightarrow z = 0$ and the equation becomes:

$$\begin{aligned} PDE: &\quad w_{zz} + w_{yy} = 0, \quad (z,y) \in R; \\ BC: &\quad w(z,0) = w(z,M) = w(0,y) = 0, w(L,y) = f(y). \end{aligned}$$

According to Case 1, the solution is:

$$w(z,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi z}{M}\right) \sin\left(\frac{n\pi y}{M}\right)$$

with

$$a_n = \frac{2}{M \sinh\left(\frac{n\pi L}{M}\right)} \int_0^M f(y) \sin\left(\frac{n\pi y}{M}\right) dy, \quad n \geq 1. \tag{44}$$

Therefore,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi(L-x)}{M}\right) \sin\left(\frac{n\pi y}{M}\right). \tag{45}$$

Case 3: Consider the boundary-value problem

$$\begin{aligned} PDE: &\quad u_{xx} + u_{yy} = 0, \quad (x,y) \in R; \\ BC: &\quad u(x,0) = u(0,y) = u(L,y) = 0, u(x,M) = f(x). \end{aligned}$$

Interchange x and y we will have Case 1. By Case 1, interchange L and M,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right), \tag{46}$$

where

$$a_n = \frac{2}{L \sinh\left(\frac{n\pi M}{L}\right)} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{47}$$

Case 4: Consider the boundary-value problem

$$\begin{aligned} PDE: &\quad u_{xx} + u_{yy} = 0, \quad (x,y) \in R; \\ BC: &\quad u(x,M) = u(0,y) = u(L,y) = 0, u(x,0) = f(x). \end{aligned}$$

This is similar to Case 2. Let $z = M - y, \quad u(x,y) = w(x,z)$. Then $y = 0 \leftrightarrow z = M; y = M \leftrightarrow z = 0$ and the equation becomes:

$$\begin{aligned} PDE: &\quad w_{xx} + w_{zz} = 0, \quad (x,z) \in R; \\ BC: &\quad w(x,0) = w(0,y) = w(L,y) = 0, w(L,M) = f(x). \end{aligned}$$

We have

$$w(x,z) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{2}{L \sinh\left(\frac{n\pi M}{L}\right)} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{48}$$

Thus

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi(M-y)}{L}\right)\sin\left(\frac{n\pi x}{L}\right). \tag{49}$$

Case 5: Consider the boundary-value problem

$$\begin{aligned} PDE: &\quad u_{xx}+u_{yy}=0, \quad (x,y)\in R;\\ BC: &\quad u(x,0)=f_1(x), u(x,M)=f_2(x), u(0,y)=f_3(y), u(L,y)=f_4(y). \end{aligned}$$

Then we just need to combine cases 1–4 together.

Theorem 14. *The solution of the above Laplace equation is:*

$$u(x,y) = u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y),$$

where

$$\begin{aligned}
u_1(x,y) &= \sum_{n=1}^{\infty} a_n \sinh\frac{n\pi(M-y)}{L}\sin\left(\frac{n\pi x}{L}\right), a_n = \frac{2}{L\sinh\left(\frac{n\pi M}{L}\right)}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx,\\
u_2(x,y) &= \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right), b_n = \frac{2}{L\sinh\left(\frac{n\pi M}{L}\right)}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx,\\
u_3(x,y) &= \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi(L-x)}{M}\sin\left(\frac{n\pi y}{M}\right), c_n = \frac{2}{M\sinh\left(\frac{n\pi L}{M}\right)}\int_0^M f_3(y)\sin\left(\frac{n\pi y}{M}\right)dy,\\
u_4(x,y) &= \sum_{n=1}^{\infty} d_n \sinh\left(\frac{n\pi x}{M}\right)\sin\left(\frac{n\pi y}{M}\right), d_n = \frac{2}{M\sinh\left(\frac{n\pi L}{M}\right)}\int_0^M f_4(y)\sin\left(\frac{n\pi y}{M}\right)dy.
\end{aligned}$$

Example Find the solution of Laplace's equation $u_{xx}+u_{yy}=0$ within

$$R = \{(x,y): 0<x<3, 0<y<2\}$$

with

$$BC: \quad u(x,0)=2x+2, u(x,2)=0, u(0,y)=2-y, u(3,y)=8-4y.$$

Solution: $L=3$, $M=2$.

$$\begin{aligned}
d_n &= \frac{2}{2\sinh\left(\frac{3n\pi}{2}\right)}\int_0^2 (8-4y)\sin\left(\frac{n\pi y}{2}\right)dy = \frac{16}{n\pi\sinh\left(\frac{3n\pi}{2}\right)},\\
c_n &= \frac{2}{2\sinh\left(\frac{3n\pi}{2}\right)}\int_0^2 (2-y)\sin\left(\frac{n\pi y}{2}\right)dy = \frac{4}{n\pi\sinh\left(\frac{3n\pi}{2}\right)},\\
a_n &= \frac{2}{3\sinh\left(\frac{2n\pi}{3}\right)}\int_0^3 (2x+2)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{-16\cos(n\pi)+4}{n\pi\sinh\left(\frac{2n\pi}{3}\right)},\\
b_n &= 0.
\end{aligned}$$

$$
\begin{aligned}
u_4(x,y) &= \sum_{n=1}^{\infty} \frac{16}{n\pi \sinh\left(\frac{3n\pi}{2}\right)} \sinh\left(\frac{n\pi x}{2}\right) \sin\left(\frac{n\pi y}{2}\right), \\
u_3(x,y) &= \sum_{n=1}^{\infty} \frac{4}{n\pi \sinh\left(\frac{3n\pi}{2}\right)} \sinh\left(\frac{n\pi(3-x)}{2}\right) \sin\left(\frac{n\pi y}{2}\right), \\
u_1(x,y) &= \sum_{n=1}^{\infty} \frac{-16\cos(n\pi)+4}{n\pi \sinh\left(\frac{2n\pi}{3}\right)} \sinh\left(\frac{n\pi(2-y)}{3}\right) \sin\left(\frac{n\pi x}{3}\right), \\
u_2(x,y) &= 0.
\end{aligned}
$$

Theorem 15. *(Polynomial solution) Consider the boundary-value problem*

$$
\begin{aligned}
PDE: &\qquad u_{xx} + u_{yy} = 0, \quad (x,y) \in R = \{(x,y) : 0 < x < L, 0 < y < M\}; \\
BC: &\qquad u(x,0) = f_1(x), u(x,M) = f_2(x), u(0,y) = f_3(y), u(L,y) = f_4(y).
\end{aligned}
$$

If all f_i are continuous and linear on the boundary, then the PDE has a polynomial solution

$$u(x,y) = ax + by + cxy + d.$$

Example Find a polynomial solution of Laplace's equation within

$$R = \{(x,y) : 0 < x < 3, 0 < y < 2\}$$

with

$$BC: \quad u(x,0) = 2x + 2, u(x,2) = 0, u(0,y) = 2 - y, u(3,y) = 8 - 4y.$$

Solution: Note that all f_i are continuous and linear on the boundary. Let

$$u(x,y) = ax + by + cxy + d.$$

$$u(x,0) = 0 \Rightarrow a = d = 2.$$

$$u(0,y) = 2 - y = by + 2 \Rightarrow b = -1.$$

$$u(3,y) = 8 - 4y = 6 - y + 3cy + 2 \Rightarrow c = -1.$$

$u(x,y) = 2x - y - xy + 2.$

4.4.2 Laplace equation inside a circle

In this case, we use polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Then we have the boundary value problem:

$$
\begin{aligned}
PDE: &\qquad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r < R; \\
BC: &\qquad u(a,\theta) = f(\theta).
\end{aligned}
$$

Theorem 16. *Let f be continuous, 2π-periodic, f' be piecewise continuous. Then the solution of the PDE above is*

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

where

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \quad n \geq 0,$$

$$b_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta)\sin(n\theta)d\theta, \quad n \geq 1.$$

Proof. Let $u(r,\theta) = F(r)G(\theta)$. We imply that

$$r^2 \frac{F''(r)}{F(r)} + r\frac{F'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} = k.$$

Thus we obtain the Euler-Cauchy equation

$$r^2 F''(r) + rF'(r) - kF(r) = 0$$

and the harmonic equation

$$G''(\theta) + kG(\theta) = 0.$$

Since $f(\theta)$ is 2π-periodic, $G(\theta)$ should be 2π-periodic. Thus $k > 0$ and $k = n^2$, $n = 1, 2, ...$ Then

$$G_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), F(r) = C_n r^n + D_n r^{-n}.$$

The eigenfunctions are

$$u_n(r,\theta) = \left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

By superposition,

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

The bounded interior solution is ($D_n = 0$ since $u(0,\theta)$ is bounded):

$$u(r,\theta) = \sum_{n=0}^{\infty} C_n r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

$$= \sum_{n=0}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right).$$

The boundary condition implies that

$$u(R,\theta) = \sum_{n=0}^{\infty} R^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) = f(\theta).$$

Thus

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \quad b_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta)\sin(n\theta)d\theta.$$

Example. Solve

$$\begin{aligned} PDE: &\qquad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r < 3; \\ BC: &\qquad u(3,\theta) = 27\cos(2\theta) - 54\sin(3\theta). \end{aligned}$$

Solution: $R = 3$. The boundary condition implies that

$$u(3,\theta) = \sum_{n=0}^{\infty} 3^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) = 27\cos(2\theta) - 54\sin(3\theta).$$

Thus $3^2 a_2 = 27, \quad 3^3 b_3 = -54$, other a_n and b_n are 0, so then

$$a_n = \begin{cases} 3, & \text{if } n = 2; \\ 0, & \text{if } n \neq 2. \end{cases}$$

$$b_n = \begin{cases} -2, & \text{if } n = 3; \\ 0, & \text{if } n \neq 3. \end{cases}$$

$$u(r,\theta) = 3r^2\cos(2\theta) - 2r^3\sin(3\theta).$$

4.4.3 Laplace equation outside a circle

Consider the boundary value problem:

$$\begin{aligned} PDE: &\qquad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r > a; \\ BC: &\qquad u(a,\theta) = f(\theta). \end{aligned}$$

Theorem 17. *Let f be continuous, 2π-periodic, f' be piecewise continuous. Let $u(r,\theta)$ be bounded in R. Then the solution of the PDE above is*

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}[a_n\cos(n\theta) + b_n\sin(n\theta)],$$

where

$$a_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \quad n \geq 0,$$

$$b_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta)\sin(n\theta)d\theta, \quad n \geq 1.$$

Example. Find the bounded solution of $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = 2$, subject to the boundary condition $u(2, \theta) = 4\cos^2(2\theta) - 5\sin(3\theta)$.

Solution: $a = 2$. The solution of the PDE above is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Note that $4\cos^2(2\theta) = 2[\cos(4\theta) + 1]$. Thus

$$u(2, \theta) = 2 + 2\cos(4\theta) - 5\sin(3\theta).$$

Therefore

$$2 + 2\cos(4\theta) - 5\sin(3\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 2^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)],$$

which implies that

$a_0 = 4, 2 = 2^{-4}a_4, a_n = 0$ for $n \neq 0, 4$; $-5 = 2^{-3}b_3$, $b_n = 0$ for $n \neq 3$. Hence

$a_0 = 4, a_4 = 32, a_n = 0$ for $n \neq 0, 4$; $b_3 = -40$, $b_n = 0$ for $n \neq 3$. Hence

$$u(r, \theta) = 2 + 32r^{-4}\cos(4\theta) - 40r^{-3}\sin(3\theta).$$

4.4.4 Laplace equation within an annulus

Consider the boundary value problem:

$$\begin{aligned} PDE: &\quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad a < r < b; \\ BC: &\quad u(a, \theta) = f(\theta), u(b, \theta) = g(\theta). \end{aligned}$$

This is a combination of the above two sections.

Theorem 18. *Let f and g be continuous, 2π-periodic, f' and g' be piecewise continuous. Then the solution of the PDE above is*

$$u(r, \theta) = \frac{a_0 + b_0 \ln r}{2} + \sum_{n=1}^{\infty}[(a_n r^n + b_n r^{-n})\cos(n\theta) + (c_n r^n + d_n r^{-n})\sin(n\theta)],$$

where

$$\begin{aligned}
a_0 + b_0 \ln a &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta, \\
a_n a^n + b_n a^{-n} &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \quad n \geq 1, \\
c_n a^n + d_n a^{-n} &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta, \quad n \geq 1, \\
a_0 + b_0 \ln b &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)d\theta, \\
a_n b^n + b_n b^{-n} &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)d\theta, \quad n \geq 1, \\
c_n b^n + d_n b^{-n} &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)d\theta, \quad n \geq 1.
\end{aligned}$$

Example. Solve

$$\begin{aligned}
PDE: &\quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < 3; \\
BC: &\quad u(1,\theta) = \cos(2\theta), u(3,\theta) = \sin(3\theta).
\end{aligned}$$

Solution: $a = 1,\ b = 3,\ f(\theta) = \cos(2\theta),\ g(\theta) = \sin(3\theta)$.

Method 1:

$$\begin{aligned}
a_0 + b_0 \ln 1 &= \frac{1}{\pi}\int_0^{2\pi} \cos(2\theta)d\theta = 0, \\
a_n 1^n + b_n 1^{-n} &= \frac{1}{\pi}\int_0^{2\pi} \cos(2\theta)\cos(n\theta)d\theta = \begin{cases} 1, & \text{if } n = 2; \\ 0, & \text{if } n \neq 2. \end{cases} \\
c_n 1^n + d_n 1^{-n} &= \frac{1}{\pi}\int_0^{2\pi} \cos(2\theta)\sin(n\theta)d\theta = 0, \quad n \geq 1, \\
a_0 + b_0 \ln 3 &= \frac{1}{\pi}\int_0^{2\pi} \sin(3\theta)d\theta = 0, \\
a_n 3^n + b_n 3^{-n} &= \frac{1}{\pi}\int_0^{2\pi} \sin(3\theta)\cos(n\theta)d\theta = 0, \quad n \geq 1, \\
c_n 3^n + d_n 3^{-n} &= \frac{1}{\pi}\int_0^{2\pi} \sin(3\theta)\sin(n\theta)d\theta = \begin{cases} 1, & \text{if } n = 3; \\ 0, & \text{if } n \neq 3. \end{cases}
\end{aligned}$$

Method 2: We start from the boundary conditions directly.

$$u(r,\theta) = \frac{a_0 + b_0 \ln r}{2} + \sum_{n=1}^{\infty}[(a_n r^n + b_n r^{-n})\cos(n\theta) + (c_n r^n + d_n r^{-n})\sin(n\theta)] \Rightarrow$$

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[(a_n + b_n)\cos(n\theta) + (c_n + d_n)\sin(n\theta)] = \cos(2\theta) \Rightarrow$$

$$\frac{a_0}{2} = 0, \quad a_2 + b_2 = 1, \quad a_n + b_n = 0 \ (n \neq 2), \quad c_n + d_n = 0 \ (n \geq 1). \tag{50}$$

Similarly,

$$u(3,\theta) = \frac{a_0 + b_0 \ln 3}{2} + \sum_{n=1}^{\infty}[(a_n 3^n + b_n 3^{-n})\cos(n\theta) + (c_n 3^n + d_n 3^{-n})\sin(n\theta)] = \sin(3\theta) \Rightarrow$$

$$\frac{a_0 + b_0 \ln 3}{2} = 0, \quad a_n 3^n + b_n 3^{-n} = 0 \ (n \neq 1), \quad c_3 3^3 + d_3 3^{-3} = 1, \quad c_n 3^n + d_n 3^{-n} = 0 \ (n \neq 3). \tag{51}$$

By (50) and (51), $a_2 = -\frac{1}{80}$, other $a_n = 0$; $b_2 = \frac{81}{80}$, other $b_n = 0$; $c_3 = \frac{27}{728}$, other $c_n = 0$; $d_3 = -\frac{27}{728}$, other $d_n = 0$. Thus

$$u(r,\theta) = \left(-\frac{1}{80}r^2 + \frac{81}{80}r^{-2}\right)\cos(2\theta) + \left(\frac{27}{728}r^3 - \frac{27}{728}r^{-3}\right)\sin(3\theta).$$

5 V Sturm-Liouville Problems

Every linear second-order ordinary differential equation of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, a_0(x) \neq 0$$

can be written in divergence form,

$$[p(x)y']' + s(x)y = 0,$$

where

$$p(x) = e^{\int a_1(x)/a_0(x)dx}, \quad s(x) = \frac{a_2(x)p(x)}{a_0(x)},$$

$p(x)$ is called an integrating factor of the differential eqn.

Example 56. *Lengendre's equation can be written as:*

$$[(1 - x^2)y']' + n(n + 1)y = 0.$$

Example 57. *Bessel's equation $z^2y_z'' + zy_z' + (z^2 - \nu^2)y = 0$ can be written as:*

$$(xy')' + \left(-\frac{\nu^2}{x} + k^2x\right)y = 0.$$

Solution: Let $z = kx$. Then

$$y_z' = y'/k, \quad y_z'' = y''/k^2.$$

Hence

$$x^2y'' + xy' + (k^2x^2 - \nu^2)y = 0.$$

Dividing by x we get the result.

5.1 Sturm-Liouville Problems

In IV we solved several initial and boundary value problems. While solving these equations we used the method separation of variables which reduces the problem to one of the following types of *Sturm-Liouville problems*

Definition 8. *Sturm-Liouville Problem is the problem which consists in finding the nonzero solutions on $[a, b]$ of the second-order ordinary differential equation (**Sturm-Liouville equation**):*

$$[p(x)y']' + [\lambda r(x) - q(x)]y = 0 \tag{52}$$

with the boundary conditions

$$k_1y(a) + k_2y'(a) = 0, (k_1, k_2) \neq (0,0) \quad l_1y(b) + l_2y'(b) = 0, (l_1, l_2) \neq (0,0). \tag{53}$$

*The values of λ for which there exist nonzero solutions are called **eigenvalues**, and the corresponding nonzero solutions are called **eigenfunctions**. The function $r(x)$ will be called the **weight function**.*

Remark 1. *If p, q, r, p' are real-valued and continuous on the interval and $p(x)$ and $r(x)$ are positive throughout that interval (or negative throughout that interval), then all the eigenvalues are real. This is important since eigenvalues are often related to frequencies, energies, or other physical quantities that must be real.*

Weight function:

Example 58. *Given the following SLP*

$$y'' + a(x)y' + \lambda b(x)y = 0, \quad c < x < d.$$

Find the weight function.

Solution: By using the integrating factor we get

$$r(x) = b(x)e^{\int a(x)dx}.$$

Example 59. *Given the following SLP*

$$y'' + 2xy' + \lambda 3y = 0, \quad c < x < d.$$

Find the weight function.

Solution:

$$r(x) = 3e^{\int 2xdx} = 3e^{x^2}.$$

Now we discuss how to get eigenvalues and eigenfunctions.

Example 60. *Find the eigenvalues and eigenfunctions:*

$$y'' + \lambda y = 0, \quad y(0) = 0, y'(L) = 0.$$

Solution: $r(x) = 1$, $q(x) = 0$, $p(x) = 1$, $a = 0$, $b = L$, $k_1 = 1$, $k_2 = 0$, $l_1 = 0$, $l_2 = 1$.

(i) When $\lambda < 0$. Set $\lambda = -\nu^2$. Then we have

$$y'' - \nu^2 y = 0.$$

The solution is $y = ce^{\nu x} + de^{-\nu x}$. The two initial conditions $y(0) = 0, y'(L) = 0$ imply that

$$\begin{cases} c + d = 0, \\ c\nu e^{\nu L} - d\nu e^{-\nu L} = 0. \end{cases}$$

We imply that $c = d = 0$, and so $y \equiv 0$, which is not an eigenfunction.

(ii) When $\lambda = 0$. Then $y'' = 0$, hence $y = cx + d$. Combining this with the two initial conditions we also get $c = d = 0$, and so, $y = 0$.

(iii) When $\lambda > 0$. Set $\lambda = \nu^2$. Then we have

$$y'' + \nu^2 y = 0.$$

The solution is $y = c \cos \nu x + d \sin \nu x$. The two initial conditions $y(0) = 0, y'(L) = 0$ imply that

$$\begin{cases} c = 0, \\ -c\nu \sin \nu L + d\nu \cos \nu L = 0. \end{cases}$$

We imply that $c = 0$, $\nu L = (k + \frac{1}{2})\pi$ $(k = 0, \pm 1, \pm 2, ...)$. Take $d = 1$. Then

$$y(x) = \sin \nu x.$$

Hence the eigenfunctions are

$$y(x) = \sin\left(\frac{2k+1}{2L}\pi x\right), \quad k = 0, 1, 2, ...$$

and the eigenvalues are

$$\lambda = \left(\frac{2k+1}{2L}\pi\right)^2, \quad k = 0, 1, 2, ...$$

Example 61. *Find the eigenvalues and eigenfunctions:*

$$(xy')' + \frac{\lambda}{x} y = 0, \quad y(1) = 0, y'(e) = 0.$$

Solution: Set $x = e^t$. Then

$$y' = \frac{dy}{dx} = e^{-t}\frac{dy}{dt},$$

which implies that

$$(xy')' = \frac{d}{dx}\left(xe^{-t}\frac{dy}{dt}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) = e^{-t}y_t''.$$

Note that, when $x = 1$, we have $t = 0$; when $x = e$, $t = 1$. Thus we obtain

$$y''_t + \lambda y = 0, \quad y(0) = 0, y'_t(1) = 0.$$

This means: $r(x) = 1$, $q(x) = 0$, $p(x) = 1$, $a = 0$, $b = 1$, $k_1 = 1$, $k_2 = 0$, $l_1 = 0$, $l_2 = 1$.

(i) When $\lambda < 0$. Set $\lambda = -\nu^2$. Then we have

$$y''_t - \nu^2 y = 0.$$

The solution is $y = ce^{\nu t} + de^{-\nu t}$. The two initial conditions $y(0) = 0, y'_t(1) = 0$ imply that

$$\begin{cases} c + d = 0, \\ c\nu e^{\nu} - d\nu e^{-\nu} = 0. \end{cases}$$

We imply that $c = d = 0$, and so $y \equiv 0$, which is not an eigenfunction.

(ii) When $\lambda = 0$. Then $y''_t = 0$, hence $y = ct + d$. Combining this with the two initial conditions we also get $c = d = 0$, and so, $y = 0$.

(iii) When $\lambda > 0$. Set $\lambda = \nu^2$. Then we have

$$y''_t + \nu^2 y = 0.$$

The solution is $y = c\cos\nu t + d\sin\nu t$. The two initial conditions $y(0) = 0, y'_t(1) = 0$ imply that

$$\begin{cases} c = 0, \\ -c\nu\sin\nu + d\nu\cos\nu = 0. \end{cases}$$

We imply that $c = 0$, $\nu = (k + \frac{1}{2})\pi$ $(k = 0, \pm 1, \pm 2, ...)$. Take $d = 1$. Then

$$y(t) = \sin\nu t.$$

Note that $t = \ln x$. Hence the eigenfunctions are

$$y(x) = \sin\left(\frac{2k+1}{2}\pi\ln x\right), \quad k = 0, 1, 2, ...$$

and the eigenvalues are

$$\lambda = \left(\frac{2k+1}{2}\pi\right)^2, \quad k = 0, 1, 2, ...$$

Example 62. *Consider the Sturm-Liouville problem*

$$y'' + 2y' + \lambda y = 0, y(0) + y'(0) = 0, y(1) = 0.$$

(a) Find all eigenvalues λ_n and corresponding eigenfunctions $y_n(x)$.

(b) Determine the weight function.

Solution: (a) The indicial (characteristic) eqn is: $r^2 + 2r + \lambda = 0$, $r = -1 \pm \sqrt{1-\lambda}$.
(i) $\lambda = 1$. Then $r = -1$ is the only root.

$$y(x) = (A + Bx)e^{-x}.$$

From the boundary conditions, $A = 0$ and $B = 0$. Thus, $y = 0$, $\lambda = 0$ is not an eigenvalue.

(ii) $\lambda < 1$. Let $\lambda = 1 - \omega^2$, $r = -1 \pm \omega$,

$$y(x) = Ae^{(-1+\omega)x} + Be^{(-1-\omega)x}.$$

From the boundary conditions, $A = 0$ and $B = 0$. Thus, $\lambda < 1$ is not an eigenvalue.
(iii) $\lambda > 1$. Now let $\lambda = 1 + \omega^2$, $r = -1 \pm i\omega$,

$$y(x) = e^{-x}(A\cos\omega x + B\sin\omega x).$$

We have

$$y(0) = A, y'(0) = -A + \omega B.$$

By $y(0) + y'(0) = 0$ we imply that $B = 0$ and $y(x) = e^{-x}A\cos\omega x$. Then by $y(1) = 0$ we have

$$\cos\omega = 0 \Rightarrow \omega = \frac{(2n+1)\pi}{2}, \Rightarrow \lambda_n = 1 + \frac{(2n+1)^2\pi^2}{4}.$$

$$y_n(x) = Ae^{-x}\cos\left(\frac{(2n+1)\pi x}{2}\right).$$

(b)

$$I(x) = e^{\int 2dx} = e^{2x}.$$

Multiply two sides of the equation by $I(x)$ we imply that

$$e^{2x}y'' + 2e^{2x}y' + \lambda e^{2x}y = 0 \Rightarrow [e^{2x}y']' + \lambda e^{2x}y = 0.$$

Thus the weight function is e^{2x}.

5.2 *Orthogonality Relations

Theorem 19. *(Orthogonality of eigenfunctions) Suppose that the functions p, q, r, r' are real-valued and continuous on the interval $a \le x \le b$ and p is positive throughout that interval. Let $y_m(x)$ and $y_n(x)$ be eigenfunctions of the Sturm-Liouville problem (52) and*

(53) corresponding to different eigenvalues λ_m and λ_n respectively. Then $y_m(x)$ and $y_n(x)$ are orthogonal on $a \le x \le b$ with respect to the weight function p:

$$(y_m, y_n) := \int_a^b p(x) y_m(x) y_n(x)\, dx = \begin{cases} 0, & \text{if } m \neq n, \\ ||y_m||^2, & \text{if } m = n, \end{cases}$$

If $r(a) = 0$ (or $r(b) = 0$) then the first boundary condition in (53) (or the second boundary condition in (53)) can be dropped from the problem.

Proof. Omitted.

5.3 Regular and Singular Sturm-Liouville Problems

Case 1. Regular problem, $p(a) \neq 0, p(b) \neq 0$. We have seen examples.

Case 2. Problem with one singular point, $p(a) = 0$, say. In this case, a single boundary condition at $x = b$, suffices, $l_1 y(b) + l_2 y'(b) = 0, (l_1, l_2) \neq (0, 0)$. Then we require that y and y' are bounded at $x = a$. A similar situation holds if $p(b) = 0$.

Example 63. *Solve the Bessel equation on $[0, R]$:*

$$[xy']' + \left(-\frac{n^2}{x} + \lambda^2 x\right) y = 0$$

with the boundary conditions

$$y(0) \text{ and } y'(0) \text{ bounded}, \quad y(R) = 0.$$

Solution: Consider the standard Bessel's equation of order n:

$$z^2 y_z'' + z y_z' + (z^2 - n^2) y = 0.$$

Let $z = \lambda x$. Then we get the given equation. One of the solution is $J_n(z) = J_n(\lambda x)$. The general solution is

$$y = c_1 J_n(\lambda x) + c_2 Y_n(\lambda x).$$

We take $c_2 = 0$ because Y_n is unbounded at the origin, so then

$$y = c_1 J_n(\lambda x).$$

$J_n(z)$ has infinitely many zeros, say $\alpha_{mn} : \alpha_{1n} < \alpha_{2n} < \cdots$. To satisfy the condition at $x = R$, let $\lambda_{mn} = \alpha_{mn}/R$. Then $J_n(\lambda_{mn}R) = J_n(\alpha_{mn}) = 0$. By Theorem 19, the sequence of the Bessel functions of the first kind $J_n(\lambda_{1n}x)$, $J_n(\lambda_{2n}x)$, $J_n(\lambda_{3n}x)$, $\cdots$ forms an orthogonal set on the interval $0 \le x \le R$ with respect to the weight $r(x) = x$.

Case 3. Problem with two singular points, $p(a) = p(b) = 0$. In this case, no boundary conditions are needed.

Example 64. *(Orthogonality of Legendre polynomials) Legendre's equation is a Sturm-Liouville equation*

$$[r(x)y']' + \lambda y = 0. \quad r(x) = 1 - x^2, \lambda = n(n+1),$$

here $p(x) = 1, q(x) = 0$.

Solution: Since $r(-1) = r(1) = 0$, we need no boundary conditions. For $n = 0, 1, 2, \cdots$, $\lambda = 0, 1 \cdot 2, 2 \cdot 3, \cdots$,the Legendre polynomials $P_n(x)$ are solutions of the problem. Hence these are the eigenfunctions. From Theorem 19, they are orthogonal, i.e.,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \quad (m \neq n).$$

Case 4. Periodic Sturm-Liouville problem: If $p(a) = p(b) \neq 0$, then (53) can be replaced by "**periodic boundary condition**"

$$y(a) = y(b), \quad y'(a) = y'(b).$$

Example 65. *Solve* $X'' + \lambda X = 0$ *with periodic boundary conditions* $X(0) = X(2\pi)$ *and* $X'(0) = X'(2\pi)$.

Solution: The eigenvalues are $\lambda_n = \left(\frac{n}{2}\right)^2$ with corresponding eigenfunctions $X_n(x) = A_n \cos(nx) + B_n \sin(nx)$ for $n = 0, 1, 2, \ldots$. Notice that $\lambda_0 = 0$ is an eigenvalue since it corresponds to $X_0(x) = A_0$ which is not necessarily zero.

Example 66. *(a) Solve the Sturm-Liouville problem:*

$$Y'' + 2Y' + (\lambda + 1)Y = 0, \quad Y(0) = 0, Y(2) = 0.$$

*(b) If

$$e^{-x} = \sum_{n=1}^{\infty} c_n Y_n(x),$$

where $Y_n(x)$ are eigenfunctions, find c_n.*

Solution: (a) The characteristic eqn is: $c^2 + 2c + (\lambda + 1) = 0$, $c = -1 \pm \sqrt{-\lambda}$.

(i) $\lambda = 0$. Then $c = -1$. $Y(x) = (A + Bx)e^{-x}$. From the boundary conditions, $A = 0$ and $B = 0$. Thus, $\lambda = 0$ is not an eigenvalue.

(ii) Now let $\lambda = -\omega^2$, $c = -1 \pm \omega$, $Y(x) = Ae^{(-1+\omega)x} + Be^{(-1-\omega)x}$. From the boundary conditions, $A = 0$ and $B = 0$. Thus, $\lambda = 0$ is not an eigenvalue.

(iii) Now let $\lambda = \omega^2$, $c = -1 \pm i\omega$, $Y(x) = e^{-x}(A\cos\omega x + B\sin\omega x)$. From the boundary conditions, $A = 0$ and $\sin 2\omega = 0$. Thus $\omega = \frac{n\pi}{2}$, and

$$Y_n(x) = e^{-x}\sin\frac{n\pi x}{2}.$$

(b) Note that $r(x) = e^{2x}$,

$$c_n = \frac{\int_a^b f(x)Y_n(x)r(x)dx}{\int_a^b Y_n^2(x)r(x)dx} = \frac{\int_0^2 e^{-x}e^{-x}\sin\frac{n\pi x}{2}e^{2x}dx}{\int_0^2\left(e^{-x}\sin\frac{n\pi x}{2}\right)^2 e^{2x}dx} = \frac{\int_0^2 \sin\frac{n\pi x}{2}dx}{\int_0^2 \frac{1}{2}[1-\cos(n\pi x)]dx}.$$

5.4 Bessel's equation

Example 67. *Bessel's equation can be written as a Sturm-Liouville equation.*

Solution: Consider the Bessel's equation

$$x^2y''(x) + xy'(x) + (\lambda^2x^2 - \nu^2)y(x) = 0.$$

Divide by x^2 on two sides,

$$y''(x) + \frac{1}{x}y'(x) + (\lambda^2 - \frac{\nu^2}{x^2})y(x) = 0.$$

Hence, by multiplying **integrating factor** $\exp\int\frac{1}{x}dx = \exp[\ln x] = x$,

$$(xy')' + \left(-\frac{\nu^2}{x} + \lambda^2x\right)y = 0,$$

here $p(x) = x$, $q(x) = -\nu^2/x$, $r(x) = x$.

We now discuss properties of Bessel's theory, not only because of their great practical importance but also as a model case for showing how properties of functions can be discovered from their series.

Propositions:

$$\begin{aligned}
\frac{d}{dx}\left[x^\nu J_\nu(x)\right] &= x^\nu J_{\nu-1}(x), \quad \nu > 0; & (54)\\
\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] &= -x^{-\nu} J_{\nu+1}(x), \quad \nu \geq 0; & (55)\\
J_{\nu-1}(x) + J_{\nu+1}(x) &= \frac{2\nu}{x} J_\nu(x), \quad \nu \geq 0; & (56)\\
J_{\nu-1}(x) - J_{\nu+1}(x) &= 2J'_\nu(x), \quad \nu \geq 0; & (57)\\
J_{1/2}(x) &= \sqrt{\frac{2}{\pi x}} \sin x; & (58)\\
J_{-1/2}(x) &= \sqrt{\frac{2}{\pi x}} \cos x. & (59)
\end{aligned}$$

Proof. Recall that

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+\nu} m! \Gamma(m+\nu+1)} x^{2m}. \tag{60}$$

Multiplying the two sides by x^ν we get

$$x^\nu J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+\nu} m! \Gamma(m+\nu+1)} x^{2m+2\nu}.$$

We differentiate this, by using recursive relation of Gamma function,

$$(x^\nu J_\nu(x))' = \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+\nu)}{2^{2m+\nu} m! \Gamma(m+\nu+1)} x^{2m+2\nu-1} = x^\nu x^{\nu-1} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+\nu-1} m! \Gamma(m+\nu)} x^{2m}.$$

This proves (54).

Similarly, multiplying the two sides of (60) by $x^{-\nu}$, then calculate derivative, then replace m by $m+1$, we can get (55).

By (54) and (55),

$$\begin{aligned}
\nu x^{\nu-1} J_\nu(x) + x^\nu J'_\nu &= x^\nu J_{\nu-1},\\
-\nu x^{-\nu-1} J_\nu(x) + x^{-\nu} J'_\nu &= -x^{-\nu} J_{\nu+1}.
\end{aligned}$$

From these two equalities we imply (56) and (57).

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. We have

$$\Gamma(m+\frac{3}{2}) = 2^{-(m+1)}(2m+1)(2m-1)\cdots 3 \cdot 1\sqrt{\pi}.$$

Now for $\nu = \frac{1}{2}$,

$$\begin{aligned} J_{1/2}(x) &= \sqrt{x}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+1/2} m!\Gamma(m+1/2+1)} = \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\Gamma(m+3/2)} \\ &= \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m! 2^{-(m+1)}(2m+1)(2m-1)\cdots 3\cdot 1\sqrt{\pi}} \\ &= \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \\ &= \sqrt{\frac{2}{\pi x}}\sin x. \end{aligned}$$

This proves (58).

Now by (54),

$$x^{1/2}J_{-1/2}(x) = \left[\sqrt{x}J_{1/2}(x)\right]' = \sqrt{\frac{2}{\pi}}\cos x.$$

This proves (59).

Example 68. *Integrate* $\int x^3 J_0(x)dx$.

Example 69. *Integrate*

$$I = \int_1^2 x^{-3} J_4(x)\, dx.$$

Solution. By (55),

$$I = -\int_1^2 \left[x^{-3}J_3(x)\right]'\, dx = -2^{-3}J_3(2) - J_3(1).$$

Now by (56) and (57),

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x), \quad J_2(x) = \frac{2}{x}J_1(x) - J_0(x).$$

We imply that

$$J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x}J_0(x). \tag{61}$$

From the table we can find the value $I = 0.0038$.

Example 70. *Represent* $J_4(x)$ *by using* $J_0(x)$ *and* $J_1(x)$.

Solution: From

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x)$$

we imply that

$$
\begin{aligned}
J_0(x) + J_2(x) &= \frac{2}{x} J_1(x), \\
J_1(x) + J_3(x) &= \frac{4}{x} J_2(x), \\
J_2(x) + J_4(x) &= \frac{6}{x} J_3(x), \\
J_3(x) + J_5(x) &= \frac{8}{x} J_4(x).
\end{aligned}
$$

Hence

$$
\begin{aligned}
J_2(x) &= \frac{2}{x} J_1(x) - J_0(x), \\
J_3(x) &= \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x), \\
J_4(x) &= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) - \left(\frac{24}{x^2} - 1\right) J_0(x).
\end{aligned}
$$

Theorem 20. *The Bessel functions of the first kind, $J_n(x)$, satisfy the following orthogonality relations:*

$$
(J_n(\lambda_{nm}x), J_n(\lambda_{nk}x)) := \int_0^R x J_n(\lambda_{nm}x) J_n(\lambda_{nk}x)\, dx = \begin{cases} 0, & \text{if } m \neq k; \\ \frac{R^2}{2} J_{n+1}^2(\lambda_{nk}R), & \text{if } m = k. \end{cases}
$$

where $\lambda_{nk} = \frac{\gamma_{nk}}{R}$ and γ_{nk} is the kth strictly positive zero of $J_n(x)$.

Example 71.

$$
\int_0^4 x J_5^2(\lambda_{5k}x)\, dx = \frac{R^2}{2} J_{n+1}^2(\lambda_{nk}R) = 8 J_6^2(\gamma_{5k}).
$$

6 VI Fourier Transform

6.1 Fourier transform and its inverse

Definition 9. *The Fourier transform of* f*:*

$$\mathcal{F}(f) := \hat{f}(\lambda) := \int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx \tag{62}$$

and the following ***inverse Fourier transform*** *of* $\hat{f}(\lambda)$*:*

$$\mathcal{F}^{-1}(\hat{f}) := f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{-i\lambda x}d\lambda. \tag{63}$$

Remark. The transform $\hat{f}(\lambda)$ can be considered as spectral density, which measures the intensity of $f(x)$ in the frequency domain.

Remark. The unitary forms of the Fourier transform are:

$$\mathcal{F}(f) := \hat{f}(\lambda) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx, \quad \mathcal{F}^{-1}(\hat{f}) := f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{i\lambda x}d\lambda.$$

Existence: If f is absolutely integrable, i.e., $\int_{-\infty}^{\infty}|f(x)|dx < \infty$, then $\mathcal{F}(f)$ exists. Similarly, if $\hat{f}$ is absolutely integrable, then $\mathcal{F}^{-1}(\hat{f})$ exists.

Example 72. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} k, & \text{if } 0 < x < a; \\ 0, & \text{otherwise,} \end{cases}$$

Solution: From (62) by integration,

$$\begin{aligned} \mathcal{F}(f) = \hat{f}(\lambda) &= \int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx \\ &= \int_0^a ke^{i\lambda x}dx = \frac{k(e^{ia\lambda}-1)}{i\lambda}. \end{aligned}$$

Example 73. *Show that*

$$\mathcal{F}(u(x-a)-u(x-b)) = \frac{e^{i\lambda b}-e^{i\lambda a}}{i\lambda}.$$

Proof: From (62) by integration,

$$\begin{aligned} \mathcal{F}(u(x-a)-u(x-b)) &= \int_{-\infty}^{\infty} (u(x-a)-u(x-b))e^{i\lambda x}dx \\ &= \int_a^b e^{i\lambda x}dx = \frac{e^{i\lambda b}-e^{i\lambda a}}{i\lambda}. \end{aligned}$$

Example 74. *Show that*

$$\mathcal{F}(e^{-|x|}) = \frac{2}{1+\lambda^2}.$$

Proof: From (62) by integration,

$$\begin{aligned}
\mathcal{F}(e^{-|x|}) &= \int_{-\infty}^{\infty} e^{-|x|} e^{i\lambda x} dx \\
&= \int_{-\infty}^{0} e^{x} e^{i\lambda x} dx + \int_{0}^{\infty} e^{-x} e^{i\lambda x} dx = \frac{1}{1+\lambda i} e^{(1+i\lambda)x}|_{-\infty}^{0} + \frac{1}{-1+\lambda i} e^{(-1+i\lambda)x}|_{0}^{\infty} \\
&= \frac{1}{1+i\lambda} - \frac{1}{-1+i\lambda} = \frac{2}{1+\lambda^2}.
\end{aligned}$$

Remark. $\lim_{x\to-\infty} e^{(1+i\lambda)x} = 0 = \lim_{x\to\infty} e^{(-1+i\lambda)x}$.

Example 75. *Show that*

$$\mathcal{F}(\delta_a) = e^{i\lambda a}.$$

Proof: Note that

$$\delta_a(f) = \int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a).$$

$$\mathcal{F}(\delta_a) = \int_{-\infty}^{\infty} \delta(x-a) e^{i\lambda x} dx = e^{i\lambda x}|_{x=a} = e^{i\lambda a}.$$

6.2 Linearity

The Fourier transform is a linear operation, that is, the following proposition holds.
Proposition (Linearity of the Fourier transform) For constants a and b, and two functions $g(x)$ and $h(x)$, we have

$$\mathcal{F}(ag + bh) = a\mathcal{F}(g) + b\mathcal{F}(h).$$

By using (62) we can easily prove it.

Theorem 21. *If $\mathcal{F}(f) := \hat{f}(\lambda)$, then for any real number α,*

$$\mathcal{F}\{f(\alpha x)\} = \frac{1}{|\alpha|}\hat{f}\left(\frac{\lambda}{\alpha}\right), \quad \mathcal{F}^{-1}\left\{\hat{f}\left(\frac{\lambda}{\alpha}\right)\right\} = |\alpha| f(\alpha x).$$

Proof. It's easy to get this by substitution $y = \alpha x$.

Example 76. *For $t > 0$, find*

$$\mathcal{F}(e^{-t|x|}).$$

Solution: Let $f(x) = e^{-|x|}$. Note that

$$\mathcal{F}(e^{-|x|}) = \mathcal{F}(f(x)) = \hat{f}(\lambda) = \frac{2}{1+\lambda^2}.$$

By the above theorem,

$$\mathcal{F}(e^{-t|x|}) = \mathcal{F}(f(tx)) = \frac{1}{t}\hat{f}\left(\frac{\lambda}{t}\right)$$
$$= \frac{1}{t}\frac{2}{1+\left(\frac{\lambda}{t}\right)^2} = \frac{2t}{t^2+\lambda^2}.$$

Example 77. *For $t > 0$, $\mathcal{F}^{-1}\left\{\frac{2t}{t^2+\lambda^2}\right\} = e^{-t|x|}$.*

Example 78. *By the definition, $\mathcal{F}^{-1}\left\{e^{-|\lambda|}\right\} = \frac{1}{\pi(1+x^2)}$.*

6.3 Shifting theorem

Theorem 22. *Suppose that f is absolutely integrable and $\mathcal{F}(f) := \hat{f}(\lambda)$. Then for any real number a,*

- *The first Shifting Theorem:*

$$\mathcal{F}\left\{e^{iax}f(x)\right\} = \hat{f}(\lambda+a), \quad \mathcal{F}^{-1}\left\{\hat{f}(\lambda+a)\right\} = e^{iax}\mathcal{F}^{-1}\left\{\hat{f}(\lambda)\right\}.$$

- *The second Shifting Theorem:*

$$\mathcal{F}\left\{f(x-a)\right\} = e^{ia\lambda}\hat{f}(\lambda), \quad \mathcal{F}^{-1}\left\{e^{ia\lambda}\hat{f}(\lambda)\right\} = f(x-a).$$

Proof. By the definition we will get both of them.

Example 79.

$$\mathcal{F}(e^{-|x+a|}) = \frac{2e^{-ia\lambda}}{1+\lambda^2}.$$
$$\mathcal{F}^{-1}\left\{\frac{e^{-ia\lambda}}{1+(\lambda-b)^2}\right\} = \frac{1}{2}e^{-bi(x+a)-|x+a|}.$$
$$\mathcal{F}(e^{iax-|x|}) = \frac{2}{1+(\lambda+a)^2}.$$

6.4 Fourier transform and derivatives

Theorem 23. *(Fourier transform of the derivative) Let $f(x)$ be continuous on the x-axis, and let $f(x) \to 0$ as $x \to \infty$. Then*

$$\mathcal{F}(f'(x)) = -i\lambda \mathcal{F}(f(x)), \tag{64}$$

$$\mathcal{F}(f''(x)) = -\lambda^2 \mathcal{F}(f(x)). \tag{65}$$

Proof. By (62), we deduce from integration by parts that

$$\begin{aligned} \mathcal{F}(f'(x)) &= \int_{-\infty}^{\infty} f'(x)e^{i\lambda x}dx \\ &= \left[f(x)e^{i\lambda x}\right]_{-\infty}^{\infty} - (i\lambda)\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx \\ &= -i\lambda \mathcal{F}(f(x)). \end{aligned}$$

This proves (64). Now by using (64) twice we imply that

$$\mathcal{F}(f''(x)) = -i\lambda \mathcal{F}(f'(x)) = (-i\lambda)^2 \mathcal{F}(f(x)) = -\lambda^2 \mathcal{F}(f(x)).$$

This proves (65).

Theorem 24. *(Derivative of Fourier transform) Suppose that $f(x)$ and $xf(x)$ are absolutely integrable and $\mathcal{F}(f) := \hat{f}(\lambda)$. Then $\hat{f}$ is differentiable and*

$$\mathcal{F}\{xf(x)\} = -i\frac{d\hat{f}(\lambda)}{d\lambda}, \quad \mathcal{F}^{-1}\left\{\frac{d\hat{f}(\lambda)}{d\lambda}\right\} = ixf(x).$$

Proof. Differentiating the two sides

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx$$

will give the result.

Example 80.

$$\mathcal{F}(e^{-tx^2}) = \frac{\sqrt{\pi}}{\sqrt{t}}e^{-\lambda^2/(4t)}, \quad \mathcal{F}^{-1}\left\{e^{-t\lambda^2}\right\} = \frac{1}{2\sqrt{t\pi}}e^{-x^2/(4t)}.$$

Proof. Let $f(x) = e^{-tx^2}$. Then $f'(x) + 2txf(x) = 0, \Rightarrow -i\lambda\hat{f}(\lambda) + 2t(-i)\hat{f}'(\lambda) = 0, \Rightarrow$

$$\frac{d\hat{f}(\lambda)}{\hat{f}(\lambda)} = \frac{\lambda}{2t}d\lambda, \Rightarrow \hat{f}(\lambda) = ce^{-\lambda^2/(4t)}, \quad c = \hat{f}(0) = \int_{-\infty}^{\infty} e^{-tx^2}dx = \frac{\sqrt{\pi}}{\sqrt{t}}.$$

Example 81. *Find the Fourier transform of the function $f(x) = xe^{-x^2}$ by using*

$$\mathcal{F}(e^{-tx^2}) = \frac{\sqrt{\pi}}{\sqrt{t}} e^{-\lambda^2/(4t)}.$$

Solution: By the linearity,

$$\begin{aligned}
\mathcal{F}\left(xe^{-x^2}\right) &= \mathcal{F}\left(-\frac{1}{2}[e^{-x^2}]'\right) = -\frac{1}{2}\mathcal{F}\left([e^{-x^2}]'\right) \\
&= -\frac{1}{2}(-i\lambda)\mathcal{F}\left(e^{-x^2}\right) = \frac{i\lambda}{2}\frac{\sqrt{\pi}}{\sqrt{1}}e^{-\lambda^2/4} \\
&= \frac{i\lambda\sqrt{\pi}}{2}e^{-\lambda^2/4}.
\end{aligned}$$

Example 82. *Find*

$$\mathcal{F}^{-1}\left\{-\frac{\sqrt{\pi}\lambda}{16}e^{-\lambda^2/16}\right\}.$$

Solution: By the above theorem,

$$\begin{aligned}
\mathcal{F}^{-1}\left\{-\frac{\sqrt{\pi}\lambda}{16}e^{-\lambda^2/16}\right\} &= \mathcal{F}^{-1}\left\{\frac{d}{d\lambda}\left(\frac{\sqrt{\pi}}{2}e^{-\lambda^2/16}\right)\right\} \\
&= ix\mathcal{F}^{-1}\left\{\frac{\sqrt{\pi}}{2}e^{-\lambda^2/16}\right\} \\
&= ixe^{-4x^2}.
\end{aligned}$$

6.5 Convolution

The convolution $f * g$ of functions f and g is defined in Part I:

$$f(x) * g(x) = (f * g)(x) := \int_{-\infty}^{\infty} f(v)g(x-v)dv = \int_{-\infty}^{\infty} f(x-v)g(v)dv.$$

Theorem 25. *(Convolution theorem) Let $f(x)$ and $g(x)$ be piecewise continuous, bounded, and absolutely integrable on the x-axis. Then*

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g), \quad (f * g)(x) = \mathcal{F}^{-1}\{\hat{f}(\lambda)\hat{g}(\lambda)\}.$$

Example 83. *Let* $f(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0, \end{cases}$ $g(x) = \begin{cases} e^{x}, & \text{if } x \leq 0; \\ 0, & \text{if } x > 0. \end{cases}$ *Find $f(x) * g(x)$, $\mathcal{F}\{f(x)\}$, $\mathcal{F}\{g(x)\}$, $\mathcal{F}\{f(x) * g(x)\}$.*

Solution: $f(x) * g(x) = \frac{1}{2}e^{-|x|}$, $\mathcal{F}\{f(x)\} = \frac{1}{1-i\lambda}$, $\mathcal{F}\{g(x)\} = \frac{1}{1+i\lambda}$, $\mathcal{F}\{f(x) * g(x)\} = \frac{1}{1+\lambda^2}$.

6.6 *Application to Heat equation on $\mathbb{R}$

Theorem 26. *Consider the initial value problem:*

$$\begin{aligned} PDE: &\quad u_{xx} = u_t, \quad -\infty < x < \infty, t > 0; \\ IC: &\quad u(x,0) = f(x), \quad -\infty < x < \infty. \end{aligned}$$

The solution is

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4t}} ds.$$

Proof. The conclusion follows from the Fourier transform of the two sides of the PDE:

$$\mathcal{F}(u_{xx}) = -\lambda^2 \hat{u}(\lambda, t), \quad \mathcal{F}(u_t) = \hat{u}_t(\lambda, t).$$

Definition 10. *The heat kernel is defined to be*

$$K(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

Remark. By using the heat kernel, we have

$$u(x,t) = f(x) * K(x,t).$$

Example 84. *Solve the heat equation:*

$$\begin{aligned} \text{PDE:} &\quad u_{xx} = u_t, \quad -\infty < x < \infty, t > 0; \\ \text{IC:} &\quad u(x,0) = \delta(x-a), \quad -\infty < x < \infty. \end{aligned}$$

Solution: By the above theorem,

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \delta(s-a) e^{-\frac{(x-s)^2}{4t}} ds \\ &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-s)^2}{4t}} \Big|_{s=a} \\ &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-a)^2}{4t}}. \end{aligned}$$

Example 85. *Solve the heat equation:*

$$\begin{aligned} \text{PDE:} &\quad u_{xx} = u_t, \quad -\infty < x < \infty, t > 0; \\ \text{IC:} &\quad u(x,0) = e^{-3x^2}, \quad -\infty < x < \infty. \end{aligned}$$

Solution: By the above theorem,

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-3s^2} e^{-\frac{(x-s)^2}{4t}}\,ds \\
&= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{12t+1}{4t}(s-\frac{x}{12t+1})^2-\frac{3x^2}{12t+1}}\,ds \\
&= \frac{e^{-\frac{3x^2}{12t+1}}}{2\sqrt{\pi t}}\sqrt{\frac{4t}{12t+1}}\int_{-\infty}^{\infty} e^{-y^2}\,dy, \quad y=\sqrt{\frac{12t+1}{4t}}(s-\frac{x}{12t+1}) \\
&= \frac{e^{-\frac{3x^2}{12t+1}}}{2\sqrt{\pi t}}\sqrt{\frac{4t}{12t+1}}\sqrt{\pi} = \frac{e^{-\frac{3x^2}{12t+1}}}{\sqrt{12t+1}}.
\end{aligned}
$$