[5pts] 1. True (T) or false (F)? *Circle the correct answer next to the statement.*

(a) If the characteristic polynomial of an $n \times n$ matrix A has fewer than n distinct roots, then A is not diagonalizable.

T F

(b) If a row echelon form of an $n \times n$ matrix A has r leading ones, then the general solution of the homogeneous system of linear equations with coefficient matrix A has r parameters.

T F

(c) If the feasible region of a linear programming problem is non-empty and bounded, then the problem has an optimal solution.

T F

(d) If X and Y are two vectors in $\mathbb{R}^n$, then $\|X+Y\| \geq \|X\| + \|Y\|$.

T F

(e) If $\{F_1, F_2, F_3\}$ is an orthogonal basis for a subspace U in $\mathbb{R}^n$, and X is a vector in U, then

$$X = proj_{F_1}(X) + proj_{F_2}(X) + proj_{F_3}(X)$$

T F

2. For each of the following two questions, write your final answer in the answer box. *You need not justify your answers, however, partial marks may be given for showing your work.*

   (a) We have a dynamical system $V_{k+1} = AV_k$ that models the growth of a certain population with two age classes. We know that the matrix A can be diagonalized as follows:

$$A = P \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} P^{-1}.$$

   [2pts] Here, P is an invertible 2×2 matrix, and λ is a real number. What can you say about the long-term growth of the population if

$\lambda > 1$: population grows without bound

$|\lambda| < 1$: population stabilizes

   (b) A baker is preparing to bake sesame-spelt bread loaves and buns for the Saturday market. She has 40 cups of flour and 100 tbsp of sesame seeds available. Each loaf contains 2 cups of flour and 3 tbsp of sesame seeds, and each bun contains $\frac{1}{2}$ cup of flour and 1 tbsp of sesame seeds. The demand is such that the number of loaves sold will be at least one third of the number of buns. If the loaves sell for \$5 a piece and the buns for \$1.50 a piece, how many loaves and how many buns should the baker make to maximize her sales revenue? ?

   [3pts] Write the above problem as a linear program in standard form. *You need not solve the problem.*

LP in standard form: maximize $p = 5x + 1.5y$

subject to

$$\begin{aligned} 2x + \tfrac{1}{2}y &\le 40 \\ 3x + y &\le 100 \\ -x + \tfrac{1}{3}y &\le 0 \\ x, y &\ge 0 \end{aligned}$$

$x \geq \frac{1}{3}y$

(c) We have a sequence of numbers $x_0, x_1, x_2, \ldots$ that satisfies the recurrence relation $x_{k+1} = 3x_k - x_{k-1}$ for $k \geq 1$. The initial terms are $x_0 = 1$ and $x_1 = 2$. Set up a dynamical system of the form $V_{k+1} = AV_k$ that will allow you to solve this recurrence relation; here $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$. That is, give the matrix A and initial vector V_0. *Do not solve the recurrence relation.*

[2pts]

$A = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$ $V_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(d) Here is a final simplex tableau for a linear program (primal problem in standard form):

x_1	x_2	x_3	x_4	x_5	M	
0	$-\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
1	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	7
0	4	0	$\frac{1}{2}$	5	1	45

[3pts] Give the optimal values of the original variables and the objective function for

(a) the primal problem: $X_1 = 7$ $M = 45$

$X_2 = 0$

$X_3 = 2$

(b) the dual problem: $y_1 = \frac{1}{2}$ $m = 45$

$y_2 = 5$

$V_{k+1} = \begin{bmatrix} X_{k+1} \\ X_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} X_k \\ X_{k+1} \end{bmatrix}$ since $X_{k+2} = 3X_{k+1} - X_k$

3. In Paradise Valley, on any given day, the weather is in one of the following three states: sunny, cloudy, or rainy. ~~On any given day~~, if it is sunny or cloudy, then it is going to be in the same state the following day with probability 0.6, and rainy with probability 0.1. However, if it is rainy on a given day, then the next day it will be rainy with probability 0.4, and is equally likely to be sunny as cloudy.

[2pts] (a) Give the probability transition matrix for the Markov process described above.

$$P = \begin{array}{c|c} \begin{bmatrix} 0.6 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.4 \end{bmatrix} & \begin{matrix} s \\ c \\ r \end{matrix} \\ \begin{matrix} s & \quad c & \quad r \end{matrix} & \end{array}$$

[2pts] (b) If the weather is sunny today, what is the probability that it will be rainy the day after tomorrow?

$$S_2 = P^2 S_0 \quad \text{with } S_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 0.6 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.4 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} * \\ * \\ 0.06 + 0.03 + 0.04 \end{bmatrix} = \begin{bmatrix} * \\ * \\ 0.13 \end{bmatrix}$$

Ans: If it is sunny today, it will be rainy with probability 0.13 the day after tomorrow.

How many days a week is

(c) In the long run, what proportion of time is the weather ~~sunny~~ rainy in Paradise Valley?
[4pts] *You may assume that this Markov process possesses a steady state vector.*

Want: S s.t. PS=S and S is a prob. vector.

0.5 * $(I-P)S=0$

0.5 * $\left[\begin{array}{ccc|c} 0.4 & -0.3 & -0.3 & 0 \\ -0.3 & 0.4 & -0.3 & 0 \\ -0.1 & -0.1 & 0.6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 4 & -3 & -3 & 0 \\ -3 & 4 & -3 & 0 \\ -1 & -1 & 6 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -6 & 0 \\ 4 & -3 & -3 & 0 \\ 3 & -4 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -6 & 0 \\ 0 & -7 & 21 & 0 \\ 0 & -7 & 21 & 0 \end{array}\right]$

1 * $\sim \left[\begin{array}{ccc|c} 1 & 1 & -6 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

* 0.5
$X_3 = t$
$X_2 = 3t$
$X_1 = 6t - 3t = 3t$

$X = t\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$

$t+3t+3t=1$
$7t=1$
* 0.5 $t=\frac{1}{7}$

steady state vector: $S = \begin{bmatrix} \frac{3}{7} \\ \frac{3}{7} \\ \frac{1}{7} \end{bmatrix}$ 0.5 *

Ans: The weather is rainy on average 1 day a week * 0.5

4. Consider the following linear program:

Maximize $M = x + 3y$

subject to
$x + y \leq 4$ (A)
$2x - y \leq 5$ (B)
$y \leq 2$ (C)
$x, y \geq 0.$ (D,E)

[3pts] (a) Graph the feasible region for this problem. *Clearly indicate the equations of the boundaries of the region, and shade the feasible region.*

0.5 ea

Halfplane (A) : $y \leq -x+4$

boundary $y = -x+4$ through $(0,4)$, $(4,0)$

Halfplane (B) : $y \geq 2x-5$

boundary $y = 2x-5$ through $(0,-5)$, $(\frac{5}{2},0)$

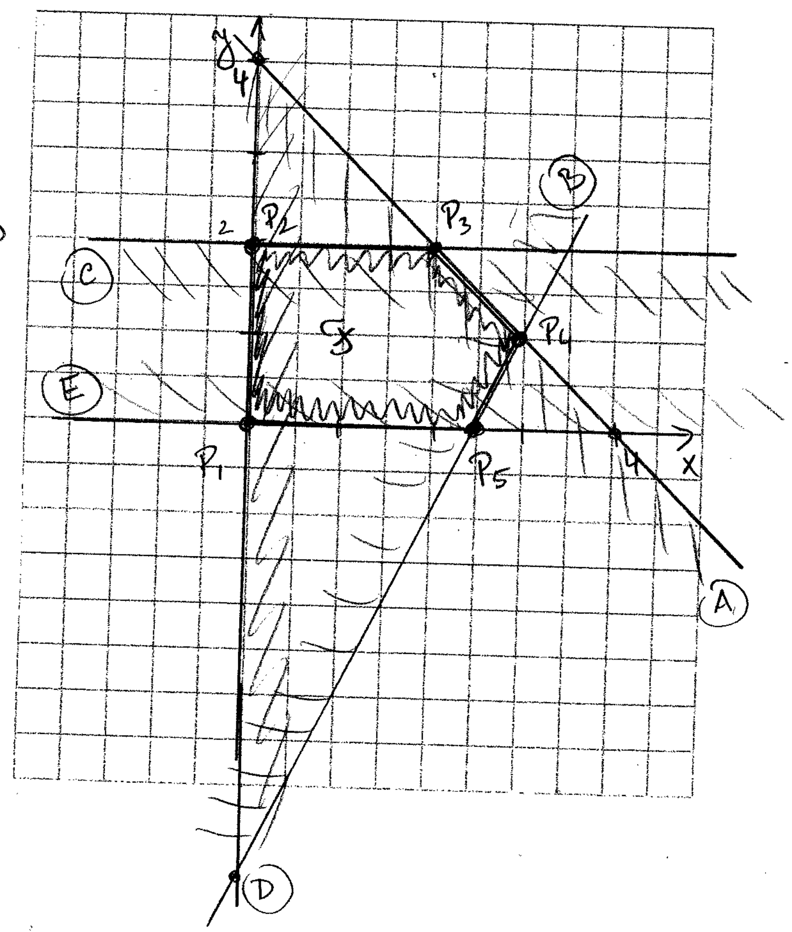
Halfplane (C) : $y \leq 2$

boundary $y = 2$

Halfplane (D) : $x \geq 0$

Halfplane (E) : $y \geq 0$

1.5



[~~2~~pts] (b) Set up the initial simplex tableau for this problem. and give the basic f. solution assoc with this tableau.

0.5 ea row
+1 BFS

x_1	x_2	x_3	x_4	x_5	M	
1	1	1	0	0	0	4
2	-1	0	1	0	0	5
0	1	0	0	1	0	2
-1	-3	0	0	0	1	0

Basic feasible solution:
$x_1 = x_2 = 0$
$x_3 = 4, \ x_4 = 5, \ x_5 = 2$

3

[~~4~~pts] (c) Solve the problem using the method of your choice.

Simplex method :

Bring in x_2 : pivot on a_{32}

1.5

x_1	x_2	x_3	x_4	x_5	M	
1	0	1	0	-1	0	2
2	0	0	1	1	0	7
0	1	0	0	1	0	2
-1	0	0	0	3	1	6

B.f.s. : $x_1 = x_5 = 0$
$x_2 = 2,$
$x_3 = 2$
$x_4 = 7$

Bring in x_3 : pivot on a_{11}

1

x_1	x_2	x_3	x_4	x_5	M	
1	0	1	0	-1	0	2
0	0	-2	1	3	0	3
0	1	0	0	1	0	2
0	0	1	0	2	1	8

Optimal solution:

$\boxed{x_1 = 2 \quad x_2 = 2}$

$(x_3 = x_5 = 0, \ x_4 = 3)$

$\boxed{M = 8}$

0.5

Geometric method:

Extreme points

i	P_i	$M_i = x_i + 3y_i$
1	$(0,0)$	0
2	$(0,2)$	6
3	$(2,2)$	8 max
4	$(3,1)$	6
5	$(\frac{5}{2},0)$	$\frac{5}{2}$

+0.5 ea pt

$-x+4=2 \Rightarrow x=2$
$y=2$

$-x+4=2x-5 \Rightarrow x=3$
$y=1$

optimal solution: $\boxed{\begin{matrix} x=2 \\ y=2 \\ M=8 \end{matrix}}$ 0.5