MAT 1341C Test1 Winter 2007

March 02, 2007. Duration: 80 minutes
Instructor: Daniel Gonçalves

Family Name: KEY !

First Name:

Student number:

Question	Response
1	
2	
3	
sub total	
4	
5	
6	
7 (Bonus)	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. You have 80 minutes to complete this exam.

2. This is a closed book exam, and no notes of any kind are allowed. **The use of calculators,** cell phones, pagers or any text storage or communication device **is not permitted.**

3. Read each question carefully -you will save yourself time and unnecessary grief later on.

4. Questions 1 to 3 are multiple choice. These questions are worth 1 points each and no part marks will be given. Please record your answers in the space provided above.

5. Questions 4 – 6 require a complete solution, and are worth 6 points each, so spend your time accordingly. Question 7 is a bonus question worth 4 points.**The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. You must answer these questions in the space provided.** Use the backs of pages if necessary.

6. Where it is possible to check your work, do so.

7. Good luck! Bonne chance!

1. Find the rank of $\begin{pmatrix} 0 & 1 & 0 & -3 & 0 \\ 1 & 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 2 & 1 & 3 & 2 & 0 \end{pmatrix}$:

- A. 0
- B. 1
- C. 2
- [✓ D. 3]
- E. 4
- F. 5

$$\begin{pmatrix} 0 & 1 & 0 & -3 & 0 \\ 1 & 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 2 & 1 & 3 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 1 & 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 2 & 1 & 3 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. The dimension of $V = \{p(x) \in P_3 : p(-x) = p(x)\}$ is equal to:

- A. 0
- B. 1
- · [✓ C. 2]
- D. 3
- E. 4
- F. 5

$P_3 = \text{span}\{1, x, x^2, x^3\}$

$V \subset P_3$

Let $p_1(x) = 1$, $p_1(-x) = 1 = p_1(x)$ ✓

$p_2(x) = x$, $p_2(-x) = -x \neq p_2(x)$

$p_3(x) = x^2$, $p_3(-x) = (-x)^2 = x^2 = p_3(x)$ ✓

$p_4(x) = x^3$, $p_4(-x) = (-x)^3 = -x^3 \neq p_3(x)$

$\therefore$ a basis for V is $\{1, x^2\} \Rightarrow \dim V = 2$

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$p(-x) = a_0 - a_1 x + a_2(-x)^2 + a_3(-x)^3 = a_0 - a_1 x + a_2 x^2 - a_3 x^3$

$p(-x) = p(x) \Rightarrow -a_1 x - a_3 x^3 = a_1 x + a_3 x^3$

$2a_1 x + 2a_3 x^3 = 0 \Rightarrow a_1 = a_3 = 0$

3. For which values of α does the vector $(5,3,\alpha)$ belong to the subspace of $\mathbf{R}^3$ spanned by $(3,2,0)$ and $(1,0,3)$?

A. $\frac{1}{2}$

B. 1

C. 2

[✓ D. $\frac{3}{2}$]

E. $\frac{5}{2}$

F. 3

$$(5,3,\alpha) = a(3,2,0) + b(1,0,3)$$

$$(5,3,\alpha) = (3a+b,\ 2a,\ 3b)$$

$$\Rightarrow \begin{cases} 3a+b = 5 & \rightarrow b = 5-3a = 5-\frac{9}{2} = 1/2 \\ 2a = 3 & \rightarrow a = 3/2 \\ \alpha = 3b & \end{cases}$$

$$\rightarrow \boxed{\alpha = 3\cdot\frac{1}{2} = \frac{3}{2}}$$

4. Suppose $a, c \in \mathbf{R}$ and consider the linear system in x, y and z:

$$\begin{array}{rcrcrcl} x & & & + & az & = & 1 \\ 2x & + & y & + & 2az & = & 3 \\ 3x & + & y & + & 4az & = & c \end{array}$$

a) If $[\,A\,|\,b\,]$ is the augmented matrix of the system above, find rank A and rank$[\,A\,|\,b\,]$ for all values of a and c.

b) Using part (a), find all values of a and c so that this system has

- (i) a unique solution,
- (ii) infinitely many solutions, or
- (iii) no solutions.

c) In case b(ii) above, give a geometric description of the set of solutions.

$$\left[\begin{array}{ccc|c} 1 & 0 & a & 1 \\ 2 & 1 & 2a & 3 \\ 3 & 1 & 4a & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & a & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & a & c-3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & a & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a & c-4 \end{array}\right]$$

a) $\text{rank } A = \begin{cases} 2 & \text{if} \quad a = 0 \\ 3 & \text{if} \quad a \neq 0 \end{cases}$

$\text{rank } [A|b] = \begin{cases} 2 & \text{if} \quad a = 0 \text{ and } c = 4 \\ 3 & \text{otherwise} \end{cases}$

b) (i) Unique sol^{n} exists $\Leftrightarrow$ rank A = rank $[A|b]$ = # variables = 3 ; hence $\underline{a \neq 0}$

(ii) ∞^{ly} many sol^{ns} $\Leftrightarrow$ rank $[A|b]$ = rank A < # variables ; $\underline{\text{hence } a = 0 \text{ and } c = 4}$

(iii) no solutions $\Leftrightarrow$ rank $[A|b]$ > rank A ; hence $\underline{a = 0,\ c \neq 4}$

c) $\left[\begin{array}{ccc|c} 1 & 0 & \overset{s}{0} & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad \begin{array}{l} x = 1 \\ y = 1 \\ z = s \end{array}$; $s \in \mathbb{R}$. This is the line through $(1, 1, 0)$ with direction $(0, 0, 1)$.

5. Consider the following functions in the vector space $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$, which are defined as follows:

$$f(x) = \sin^2 x, \quad g(x) = \cos^2(x) \text{ and } h(x) = 2 - 3\cos^2(x).$$

Define a subspace U of $\mathbf{F}(\mathbf{R})$ by

$$U = span\{f, g\}.$$

a) Show that $\{f, g\}$ is linearly independent.

b) Find $\dim U$.

c) Show that $h \in U$

d) What is the dimension of $span\{f, g, h\}$?

(a) Suppose that there exists $a, b \in \mathbb{R}$ such that $af(x) + bg(x) = 0 \ \forall x \in$

ie, $a \sin^2 x + b \cos^2 x = 0 \quad \forall x \in \mathbb{R}$.

$$\left.\begin{array}{lll} \text{at } x = 0, & a.0 + b.1 = 0 & \Rightarrow b = 0 \\ x = \pi/2, & a.1 + b.0 = 0 & \Rightarrow a = 0 \end{array}\right\} \Rightarrow a = b = 0 \Rightarrow \{f, g\} \text{ is l.i}$$

(b) Since $\{f, g\}$ is l.i and $\{f, g\}$ spans U, we have that $\dim U = 2$

(c)
$$h(x) = 2.1 - 3.\cos^2 x$$
$$= 2(\underbrace{\sin^2 x + \cos^2 x}_{=1}) - 3.\cos^2 x$$
$$= 2\sin^2 x - \cos^2 x = 2.f(x) + (-1)g(x)$$

(d) Since $h \in span\{f, g\}$, we have that $span\{f, g, h\} = span\{f, g\}$.
Therefore $\dim(\{f, g, h\}) = \underline{2}$

$1+\frac{1}{2}$ each ..

6. State whether the following are true or false, in the box after the statement. You must justify your answer: if true, explain why, if not, give an example to show it is false.

$\underset{\text{correct}}{\frac{1}{2}} + \underset{\text{partial just}^n}{\frac{1}{2}} + \frac{1}{2}$ good just^n

a) If V is a vector space with u, v, w in V, and $\{u, v, w\}$ spans V, then $\{u, v, w\}$ is also linearly independent. $\boxed{\text{False}}$

e.g. $V = \mathbb{R}^2$

$u = (1, 0)$

$v = (0, 1)$

$w = (0, 0)$.

b) If U is a vector space with u, v, w in U, and $\{u, v, w\}$ is linearly dependent, then $w \in \operatorname{span}\{u, v\}$. $\boxed{\text{False}}$

$U = \mathbb{R}^2$

$u = (0, 0) = v$

$w = (1, 0)$.

(or any correct just^n)

c) Every linear system of 3 equations in 4 unknowns has infinitely many solutions. FALSE

$$\left.\begin{array}{r} x+y+z+w=0 \\ x+y+z+w=1 \\ z+w=-2 \end{array}\right\rangle \text{NO SOLUTION!}$$

d) Every homogeneous linear system of 333 equations in 332 unknowns is consistent. TRUE

$\vec{0}$ is always a solution of a homogeneous system.

7. [Bonus] Let S_3 be the vector space of all 3x3 anti-symmetric matrices, that is,

$$S_3 = \{A \in M_{3x3} : A = -A^T\}.$$

a) Find a basis of S^3 and hence give its dimension.

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Then $-A^T = \begin{bmatrix} -a & -d & -g \\ -b & -e & -h \\ -e & -f & -i \end{bmatrix}$. From $A = -A^T$ we have

$a = -a$, $e = -e$, $i = -i \Rightarrow a = e = i = 0$

$b = -d$, $c = -g$, $f = -h$,

Hence, A is anti-symmetric if $A = \begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$\left\{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right\}$ is l.i. and spans S_3.

$\dim S_3 = 3$

b) Is it possible to find a basis for the subspace of S_3 given by $U = \{A \in S_3 \mid A = A^T\}$?

No!

$U = \{A \in S_3 / A = A^T\} = \{A \in M_{3x3} : A = -A^T \ \underline{\text{and}}\ A = A^T\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$ is l.d.