MAT 1341C Test 1 Solutions.

1. Which of the following statements are true?

I. A set $\{u, v, w\}$ of vectors is linearly independent iff for scalars $a, b, c \in \mathbf{R}$, $au + bv + cw = 0$ implies $a = b = c = 0$. TRUE

II. A set $\{u, v, w\}$ of vectors is linearly independent iff for scalars $a, b, c \in \mathbf{R}$, $au + bv + cw = 0$ when $a = b = c = 0$. FALSE

III. A set $\{u, v, w\}$ of vectors is linearly independent iff u is not a linear combination of v and w.

FALSE: $\{\underset{u}{(1,0)}, \underset{v}{(0,1)}, \underset{w}{(0,0)}\}$ is dependent but $u \notin \operatorname{span}\{v, w\}$

IV. $\{(-1, 1), (1, 2)\}$ spans $\mathbf{R}^2$.

TRUE: they are non-zero & non parallel

V. $\{\underset{u}{(1, 0, 1)}, \underset{v}{(0, 1, 1)}, \underset{w}{(1, 1, 2)}\}$ spans $\mathbf{R}^3$.

(This is false; because at most two are true by the answers; indeed $(1,0,0) \notin \operatorname{span}\{u,v,w\}$.)

A. II & IV
B. I & II
C. I & IV
D. III & V
E. III & II
F. I & V

2. Which of the following are subspaces of $\mathbf{M}_{2\,2}$?

$$U = \left\{ \begin{bmatrix} x & x \\ y & x+y \end{bmatrix} \in \mathbf{M}_{2\,2} \mid x,\ y \in \mathbf{R} \right\}, = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

$\therefore$ is a s.s.

$$V = \left\{ \begin{bmatrix} x & x+y \\ 3y & y \end{bmatrix} \in \mathbf{M}_{2\,2} \mid x,\ y \in \mathbf{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} \right\}$$

$\therefore$ is a s.s

$$W = \left\{ \begin{bmatrix} x & x \\ xy & y \end{bmatrix} \in \mathbf{M}_{2\,2} \mid x,\ y \in \mathbf{R} \right\}$$

IS not

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in W$ $(x=1, y=0)$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$ $(x=0, y=1)$

but +

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \notin W$ (since x must be 1, y " 1, and then $xy = 1 \neq 0$)

A. U only
B. V only
C. W only
D. U and V only
E. U and W only
F. V and W only

3. Which two of the following statements are true?

I. The span of two distinct vectors u and v in $\mathbf{R}^3$ is a plane through the origin.

II. The span of a single vector u in $\mathbf{R}^2$ is a line.

III. A set of vectors $\{u, v, w\} \subseteq X$ spans a vector space X if every $x \in X$ is a linear combination of v and w.

IV. Any spanning set for $\mathbf{R}^2$ contains at least two elements.

V. $\left\{\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{B}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \underset{C}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}\right\}$ spans $\mathbf{M}_{22}$.

A. I & III
B. II & IV
C. I & II
D. III & IV
E. III & II
F. I & V

(I) — False e.g. Span $\{(0,0,0), (1,0,0)\}$ is the x-axis
(or span $\{(1,0,0), (2,0,0)\}$ is the x-axis)

(II) — False, since the vector could be zero &
Span $\{(0,0)\} = \{(0,0)\}$ is a point.

(III) TRUE, Since if $x \in$ span $\{v, w\}$, and $u \in X$,
then $x = av + bw \Rightarrow x = av + bw + 0u$
$\Rightarrow x \in$ span $\{u, v, w\}$.

(IV) TRUE, Since we know that span$\{v\}$ is either
$\{0\}$ (if $v = 0$) or a line through the origin with dirn
v if $v \neq 0$.

(V) FALSE e.g. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = aA + bB + cC \Rightarrow a = 0,\ b = 1\ \&\ b = 0$, a contradiction.
($\&\ c = 0$)

4. Let $v = (0,1,1)$ and $U = \{u \in \mathbf{R}^3 \mid \text{proj}_v\, u = 0\}$

$\frac{1}{2}$ a) Show that if $u = (x,y,z) \in \mathbf{R}^3$, then $\text{proj}_v\, u = \dfrac{y+z}{2}(0,1,1)$

b) Find a Cartesian equation for U, i.e., find $a,b,c,d \in \mathbf{R}$ such that

$\frac{1}{2} + \frac{1}{2}$

$$U = \{(x,y,z) \in \mathbf{R}^3 \mid ax + by + cz = d\},$$

$1\frac{1}{2}$ Give a complete geometric description of U.

1 c) Is U a subspace of $\mathbf{R}^3$? $\frac{1}{2} + \frac{1}{2}$

2 d) Find a spanning set for U. $1 + 1$

a) $\text{proj}_v\,(x,y,z) = \left[\dfrac{(x,y,z)\cdot(0,1,1)}{\|(0,1,1)\|^2}\right](0,1,1) = \left(\dfrac{y+z}{2}\right)(0,1,1)$

b) Note that $\text{proj}_v(x,y,z) = (0,0,0) \Leftrightarrow \left(\frac{y+z}{2}\right)(0,1,1) = 0$

$\Leftrightarrow y + z = 0$. (Since $(0,1,1) \neq (0,0,0)$).

Hence $U = \{(x,y,z) \mid y+z=0\}$.

So U is the plane through the origin with normal $(0,1,1)$. $(= v.)$

c) Since U is a plane through the origin in $\mathbb{R}^3$, U is a subspace of $\mathbb{R}^3$.

d) Note that $U = \{(x,y,z) \in \mathbb{R}^3 \mid y+z=0\}$

$= \{(x,-z,z) \in \mathbb{R}^3 \mid x, z \in \mathbb{R}\}$

$= \text{span}\{(1,0,0), (0,-1,1)\}$

$\therefore \{(1,0,0),(0,-1,1)\}$ spans U.

4. Let $v = (0, 1, -1)$ and $U = \{u \in \mathbf{R}^3 \mid \operatorname{proj}_v u = 0\}$

$\frac{1}{2}$ a) Show that if $u = (x, y, z) \in \mathbf{R}^3$, then $\operatorname{proj}_v u = \frac{y-z}{2}(0, 1, -1)$

$2\frac{1}{2}$ b) Find a Cartesian equation for U, i.e., find $a, b, c, d \in \mathbf{R}$ such that

$$U = \{(x, y, z) \in \mathbf{R}^3 \mid ax + by + cz = d\}, \quad \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)$$

Give a complete geometric description of U. $\left(1\frac{1}{2}\right) = 3 \times \frac{1}{2}$.

1 c) Is U a subspace of $\mathbf{R}^3$? $\left(\tfrac{1}{2} + \tfrac{1}{2}\right)$

2 d) Find a spanning set for U. $(1) + (1)$

a) $\operatorname{proj}_v (x, y, z) = \left[\dfrac{(0,1,-1)\cdot(x,y,z)}{\|(0,1,-1)\|^2}\right](0,1,-1) = \left(\dfrac{y-z}{2}\right)(0,1,-1)$

b) $\operatorname{proj}_v (x,y,z) = (0,0,0) \Leftrightarrow \left(\frac{y-z}{2}\right)(0,1,-1) = (0,0,0) \overset{*}{\Leftrightarrow} y - z = 0.$

$\therefore \quad U = \{(x,y,z) \mid y - z = 0\} \qquad (a = 0,\ b = 1,\ c = -1)$

[(*) Because $(0,1,-1) \neq (0,0,0)$] (See (c).)

c) Yes, because it is a plane through the origin in $\mathbb{R}^3$ with normal $(0,1,-1)$ $(= v.)$

d) Now $(x,y,z) \in U \Leftrightarrow (x,y,z) = (x,z,z) \Leftrightarrow$
$(x,y,z) = x(1,0,0) + z(0,1,1)$, for some $x, z \in \mathbb{R}$

Hence $U = \operatorname{span}\{(1,0,0), (0,1,1)\}$.

5. Let $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$ be the vector space of real-valued functions defined on $\mathbf{R}$. Recall that the zero of $\mathbf{F}(\mathbf{R})$ is the function that has the value 0 for all $x \in \mathbf{R}$.

Define three functions in $\mathbf{F}(\mathbf{R})$ by

$$f(x) = 1 + x,$$
$$g(x) = x + x^2,$$
$$h(x) = x + x^2 + x^3,$$

and let $W = \text{span}\{f, g, h\}$.

3 a) Show, using the definition of linear independence, that $\{f, g, h\}$ is linearly independent.

$1\frac{1}{2}$ b) If $j(x) = 1 - x^2 + x^3$ show that $j \in W$.

$1\frac{1}{2}$ c) Is $W = \text{span}\{f, g, h, j\}$? Explain your answer.

a) Suppose $af + bg + ch = 0$ (zero function.) Then

$$\underbrace{a(1+x) + b(x+x^2) + c(x+x^2+x^3) = 0}_{(*)}, \quad \forall x \in \mathbb{R}$$

(1)

ie. $a + (a+b+c)x + (b+c)x^2 + cx^3 = 0$.

At $x=0$, we have $a = 0$

$x=1$ " $2b + 3c = 0$

$x=-1$ " $-c = 0$

(Assume this now)

(1) Some set of equation from (*)

(1) $\Rightarrow a=b=c=0$

$\therefore a = b = c = 0$ and so $\{f, g, h\}$ is l.i..

b) Since $f - g = 1 - x^2$ and $h - g = x^3$,

$j = 1 - x^2 + x^3 = f - g + h - g = f + h - 2g$

Hence $j \in \text{span}\{f, g, h\} = W$.

(1) knowing what to do

($\frac{1}{2}$) correct soln

c) Yes: Since (1) $j \in W$ by (b),

($\frac{1}{2}$) $\text{span}\{f, g, h, j\} = \text{span}\{f, g, h\} = W$.

6. State whether the following are true (always), or may be false, in the box after the statement. You must justify your answer: if true, explain why, if not, give an example to show it is false.

3 a) If V is a vector space with u, v, w in V, and $\{u, v, w\}$ is linearly independent, then $\{u, v\}$ is also linearly independent.

Suppose $au + bv = 0$. (For scalar a, b) $\boxed{\text{TRUE}}$

Then $au + bv + 0w = 0$. But $\{u, v, w\}$ is l.i., so

$a = b = (0 =) 0$ ie. $a = b = 0$

Hence $\{u, v\}$ is l.i.

(a) & (b) 1 - correct answer.

2 - justn/example

3 b) If W is a vector space with u, v, w in W, and $\{u, v, w\}$ spans W, then $\{u, v\}$ also spans W.

e.g. $W = \mathbb{R}^3$, $u = (1,0,0)$, $v = (0,1,0)$, $w = (0,0,1)$ $\boxed{\text{FALSE}}$

Then $\{u, v, w\}$ span $\mathbb{R}^3$ but $\{u, v\}$ spans only the x-y plane.

or $W = \mathbb{R}^2$, $u = (0,0)$, $v = (1,0)$, $w = (0,1)$. Then span $\{(0,0), (1,0), (0,1)\} =$ span $\{(1,0), (0,1)\} = \mathbb{R}^2$ but span $\{(0,0), (1,0)\}$ is only the x-axis.

7. [Bonus] Suppose that u, v, w are non-zero vectors in $\mathbf{R}^4$ such that $u \cdot v = u \cdot w = v \cdot w = 0$. Prove that $\{u, v, w\}$ is linearly independent.

(*) Suppose $au + bv + cw = 0$ for scalars a, b, c.

Taking dot products of both sides of this equation with u, v and w, we obtain:

$$u \cdot (au + bv + cw) = a(u \cdot u) + b(u \cdot v) + c(u \cdot w) = 0$$
$$v \cdot (au + bv + cw) = a(v \cdot u) + b(v \cdot v) + c(v \cdot w) = 0$$
$$\& \ w \cdot (au + bv + cw) = a(w \cdot u) + b(w \cdot v) + c(w \cdot w) = 0$$

But $u \cdot v = u \cdot w = v \cdot w = 0$, so we obtain

$$\left.\begin{aligned} a\|u\|^2 &= 0 \\ b\|v\|^2 &= 0 \\ c\|w\|^2 &= 0 \end{aligned}\right\} (*)$$

But u, v, w are non-zero, and so $\|u\|^2 \neq 0$, $\|v\|^2 \neq 0$ & $\|w\|^2 \neq 0$. Hence $(*)$ implies that $a = b = c = 0$. Thus $\{u, v, w\}$ is l.i.

1 (*) + 1 dot products + 1 $x \neq 0 \Rightarrow \|x\|^2 \neq 0$

+ 1 well-written