Question 1. [4 points] Compute the derivatives of the following functions

(a) $f(x) = e^{(10x+12)} \ln(x^{2010} + 1)$.

$$f'(x) = \left(e^{10x+12}\right)' \ln(x^{2010}+1) +$$

$$+ e^{(10x+12)} \cdot \left(\ln(x^{2010}+1)\right)' =$$

$$= e^{(10x+12)} \cdot 10 \cdot \ln(x^{2010}+1) +$$

$$+ e^{(10x+12)} \cdot \frac{1}{x^{2010}+1} \cdot 2010 \cdot x^{2009} =$$

$$= e^{(10x+12)} \left[10 \cdot \ln(x^{2010}+1) + \frac{2010\, x^{2009}}{x^{2010}+1}\right]$$

$f'(x) =$ $\boxed{e^{(10x+12)} \left[10 \cdot \ln(x^{2010}+1) + \frac{2010\, x^{2009}}{x^{2010}+1}\right]}$

(b) $g(x) = \cos^2(\sqrt{x})$.

$$g'(x) = 2\cos(\sqrt{x}) \cdot \left(-\sin(\sqrt{x})\right) \cdot \frac{1}{2\sqrt{x}} =$$

$g'(x) =$ $\boxed{-\frac{1}{\sqrt{x}} \cdot \cos(\sqrt{x}) \sin(\sqrt{x})}$

Question 2. [8 points] Determine if the following limits exist. If the limit exists, compute the limit without using table of values.

(a) $\lim_{x\to 2} \frac{x^2+x-6}{x^2-2x+4} = \boxed{\frac{0}{4} = 0}$

$\frac{x^2+x-6}{x^2-2x+4}$ is continuous at $x=2$.

(b) $\lim_{x\to 0} \frac{x}{1-\sqrt{1+x}} = \boxed{}$

$\left[\frac{0}{0}\right]$

$\lim_{x\to 0} \frac{x(1+\sqrt{1+x})}{1-(1+x)} = \lim_{x\to 0} -\frac{x(1+\sqrt{1+x})}{x} = -\lim_{x\to 0}(1+\sqrt{1+x}) = -2$ or by L'Hospital's rule:

$\lim_{x\to 0} \frac{x}{1-\sqrt{1+x}} = \lim_{x\to 0} \frac{1}{-\frac{1}{2\sqrt{1+x}}} = -2$

(c) $\lim_{x\to 0^+} x\ln x = \boxed{}$

$[0\cdot -\infty]$

$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \left[\frac{-\infty}{\infty}\right] = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} =$

$= \lim_{x\to 0^+} \frac{1}{x}\cdot\left(-\frac{x^2}{1}\right) = -\lim_{x\to 0^+} x = 0.$

(d) $\lim_{x\to\infty} \frac{x^2}{e^x} = \boxed{}$

$\left[\frac{\infty}{\infty}\right]$

$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{L.R}{=} \lim_{x\to\infty} \frac{2x}{e^x} = \left[\frac{\infty}{\infty}\right] \overset{L.R.}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$

Question 3. [2 points] A golf ball hit with an angle of θ radians and initial velocity of 10m/s will fly for a distance of $d(\theta) = 20.41\sin(\theta)\cos(\theta)$ metres before it lands (neglecting air resistance). Find the angle θ^* between 0 and $\pi/2$ radians that maximizes the distance flown, and find the maximal distance.

$d(\theta) = 20.41\sin(\theta)\cos(\theta)$

$d'(\theta) = 20.41[\sin'(\theta)\cdot\cos(\theta) + \sin(\theta)\cdot\cos'(\theta)] =$

$= 20.41[\cos^2(\theta) - \sin^2(\theta)] = 20.41[\cos^2\theta - (1-\cos^2\theta)] =$

$= 20.41[2\cos^2(\theta) - 1] = 0$

$2\cos^2(\theta) = 1$

$\cos^2(\theta) = \frac{1}{2}$

$\cos(\theta) = \pm\frac{1}{\sqrt{2}}$, since $\theta \in [0, \frac{\pi}{2}]$, we exclude $\cos(\theta) = -\frac{1}{\sqrt{2}}$ ($\theta = \frac{3\pi}{4}$), Thus $\theta = \frac{\pi}{4}$ ($\cos(\theta) = \frac{1}{\sqrt{2}}$).

Answer: $\theta^* = \boxed{\frac{\pi}{4}}$, $d(\theta^*) = \boxed{20.41 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}}$. $= \frac{20.41}{2}$

! [$d(0) = 0$; $d(\frac{\pi}{2}) = 0$.] by Extreme Value Theorem. $d(\theta^*) = \nearrow$.

Question 4. [4 points] Find the Taylor polynomial of degree 3 for $f(x) = \sin(2x) + x^2$ with base point $a = 0$. (Note that x is in radians!)

$P_{3,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \frac{f'''(a)(x-a)^3}{6}$

$f(a) = f(0) = \sin(0) + 0^2 = 0$

$f'(x) = 2\cos(2x) + 2x$

$f'(0) = 2\cos(0) + 0 = 2$

$f''(x) = -4\sin(2x) + 2$; $f''(0) = 2$

$f'''(x) = -8\cos(2x)$; $f'''(0) = -8$

(a) $P_3(x) = \boxed{0 + 2\cdot x + \frac{2x^2}{2} + \frac{(-8)x^3}{6} = 2x + x^2 - \frac{4x^3}{3}}$

$\parallel$

$P_{3,a}(x)$

$f(-0.1) = \sin(2\cdot(-0.1)) + (-0.1)^2 = -0.188669$

(b) Use this polynomial to approximate $f(-0.1)$. $\hat{f}(-0.1) = \boxed{}$

$P_3(0.1) = 2(-0.1) + 0.01 - \frac{4}{3}\cdot(-0.001) = -0.188666$

Question 5. [4 points]
(a) Give the definition for the function $f(x)$ to be differentiable at the point $x = a$.

$$\boxed{\text{If } \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ exists}}$$

(b) Use the definition (first principles) to find the derivative of

$$f(x) = \frac{2x}{x+1}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(h)}{h} = $$

$$= \lim_{h\to 0} \frac{\frac{2(x+h)}{x+h+1} - \frac{2x}{x+1}}{h} = \lim_{h\to 0} \frac{2(x+h)(x+1)-2x(x+h+1)}{h(x+h+1)(x+1)} =$$

$$= \lim_{h\to 0} \frac{\cancel{2x^2}+\cancel{2xh}\cancel{+2x}+2h-\cancel{2x^2}-\cancel{2xh}-\cancel{2x}}{\cancel{h}(x+h+1)(x+1)} =$$

$$= \lim_{h\to 0} \frac{2}{(x+h+1)(x+1)} = \frac{2}{(x+1)^2}$$

Question 6. [5 points] Consider the DTDS

$$x_{t+1} = \frac{1+x_t}{1+x_t^2}, \qquad t = 0, 1, 2, \ldots.$$

(a) The updating function of this DTDS is $f(x) =$ $\boxed{\dfrac{1+X}{1+X^2}}$

(b) The only positive equilibrium is $x^* =$ $\boxed{1}$

$f(X) = X$

$\dfrac{1+X}{1+X^2} = X$

$1+X = X + X^3$

$X^3 = 1$

$X = 1 \quad \Rightarrow \quad \boxed{X^* = 1}$

(c) According to the derivative test, is the equilibrium stable or unstable? Answer: $\boxed{\text{Stable}}$

$$f'(X) = \frac{(1+X)' \cdot (1+X^2) - (1+X)\cdot(1+X^2)'}{(1+X^2)^2} =$$

$$= \frac{1+X^2 - 2X(1+X)}{(1+X^2)^2} = \frac{1+X^2-2X-2X^2}{(1+X^2)^2} = \frac{1-2X-X^2}{(1+X^2)^2}$$

$$|f'(1)| = \left|\frac{1-2-1}{(1+1)^2}\right| = \left|-\frac{2}{4}\right| = \frac{1}{2} < 1$$

(d) Starting from $x_0 = 5$, calculate x_1, x_2, x_3. Answer: $\boxed{}$

$$X_1 = \frac{1+X_0}{1+X_0^2} = \frac{1+5}{1+25} = \frac{6}{26} = 0.23076$$

$$X_2 = \frac{1+X_1}{1+X_1^2} = \frac{1+0.23076}{1+(0.23076)^2} = 1.168536$$

and so on . . .

Question 7. [12 points] Compute the following indefinite integrals:

(a) $\int (x^3 + \cos x)dx = \boxed{\int x^3 dx + \int \cos x \, dx =}$

$= \frac{x^4}{4} + \sin x + C$

(b) $\int \frac{e^{2x}+4}{e^{2x}} dx = \boxed{\int (1 + 4e^{-2x})dx = \int dx} + 4\int e^{-2x} dx =$

$= x + \frac{4e^{-2x}}{-2} + C = x - 2e^{-2x} + C.$

(c) $\int 16x^3 \ln(7x) dx = \boxed{4x^4 \ln(7x) - x^4 + C}$

In our case

$\int u\,dv = uv - \int v\,du = \ln(7x) \cdot 4x^4 - \int 4x^4 \frac{dx}{x} = 4x^4 \ln(7x) - 4\int x^3 dx =$

$\int x^3 dx = \frac{x^4}{4}$

$\ln(7x) = u$

$\frac{1}{7x} \cdot 7dx = du \Rightarrow \frac{dx}{x} = du$

$16x^3 dx = dv = v' dx$

$v' = 16x^3; \; v = 16\int x^3 dx = \frac{16x^4}{4} = 4x^4$

(d) $\int \frac{\sin(\frac{1}{x})}{x^2} dx = \boxed{\cos\left(\frac{1}{x}\right) + C}$

$\frac{1}{x} = u(x)$

$-\frac{1}{x^2} dx = du \Rightarrow \frac{dx}{x^2} = -du$

$\int \frac{\sin(\frac{1}{x})}{x^2} dx = -\int \sin(u) du = +\cos(u) + C =$

$+\cos\left(\frac{1}{x}\right) + C$

Question 8. [4 points] Consider the function $f(x) = e^{-x} - x$.
(a) [2 points] Explain why this function has a zero in the interval $[0, 1]$.

$f(0) = e^0 - 0 = 1 > 0$
$f(1) = e^{-1} - 1 = \frac{1}{e} - 1 < 0$
using Intermediate Value Theorem, we conclude that there exists a root c in $(0,1)$ (or $f(c)=0$ on $(0,1)$)

(b) [2 points] Calculate the zero to 3 decimal places, i.e. $f(x^*) = 0$ for $x^* =$ $\boxed{}$

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}, \quad f'(X_n) \neq 0$$

$f(x) = e^{-x} - x$
$f'(x) = -e^{-x} - 1 = -(e^{-x}+1)$

$$X_{n+1} = X_n - \frac{e^{-X_n} - X_n}{-(e^{-X_n}+1)} = X_n + \frac{e^{-X_n} - X_n}{e^{-X_n}+1}$$

Take $X_0 = 0.5 \in (0,1)$

$$X_1 = 0.5 + \frac{e^{-0.5} - 0.5}{e^{-0.5}+1} \simeq 0.566311$$

we can do one more iteration

$$X_2 = 0.566311 + \frac{e^{-0.566311} - 0.566311}{e^{-0.566311}+1} \simeq 0.5671431$$

Question 9. [7 points] An athlete starts a marathon with a speed of 14 km/h. Due to an existing ankle injury, she is forced to slow down and her speed decreases at a constant rate according to the differential equation $v'(t) = -2$ for $t \geq 0$ with $v(0) = 14$. Answer the following questions.

(a) Find the equation for the speed (in km/h) as a function of time (in hours).

$v(t) = \boxed{\int -2\,dt = -2t + C}$, where C is a constant; it can be found from the initial condition $v(0) = 14$

$v(t) = -2t + C$

$v(0) = -2 \cdot 0 + C = 14 \quad \Rightarrow \boxed{C = 14}$

$v(t) = -2t + 14$; $v(t) \geq 0$ when $-2t + 14 \geq 0$

$2t \leq 14$

$t \leq 7$ hours

velocity is negative is $t > 7$ (does not make sens to talk about it after 7 hou

(b) Solve the pure-time differential equation $\frac{dp}{dt} = v(t)$, $p(0) = 0$ for the location p (in km).

$p(t) = \boxed{\int v\,dt = \int (-2t + 14)\,dt} = -t^2 + 14t + C$

$p(t) = -t^2 + 14t + C$

$p(0) = 0 \quad \Rightarrow \boxed{C = 0}$

$p(t) = -t^2 + 14t$

(c) How long will it take the athlete to complete the marathon (42 km)?

Finish time $T^* = \boxed{t_2 \approx 4.4 \text{ hours}}$

$p(t) = -t^2 + 14t = 42$

$t^2 - 14t + 42 = 0$

$D = 196 - 4 \cdot 42 = 28$

$t_1 = \frac{14 + \sqrt{28}}{2} \approx 9.6 \text{(hours)} \leftarrow$ discard this ~~value~~ ($t \leq 7$)

$t_2 = \frac{14 - \sqrt{28}}{2} \approx 4.4 \text{(hours)}$

$T^* = t_2$

What is her speed when crossing the finishing line? $v(T^*) =$

$v(T^*) = -2T^* + 14 = -2\left(\frac{14 - \sqrt{28}}{2}\right) + 14 = \sqrt{28}$ km/h.

Question 10. [10 points] Consider the function $f(x) = \frac{1}{x^2} + \frac{1}{2x^3}$. Follow these steps to graph the function.

(a) The domain of f is $\boxed{\{x \in \mathbb{R},\ x \neq 0\}}$

$$f(x) = \frac{2x+1}{2x^3}$$

(b) The x-intercept(s) of f are $\boxed{-\frac{1}{2}}$

$$\frac{2x+1}{2x^3} = 0$$

$$x = -\frac{1}{2}$$

(c) The derivative of f is $f' = \boxed{}$

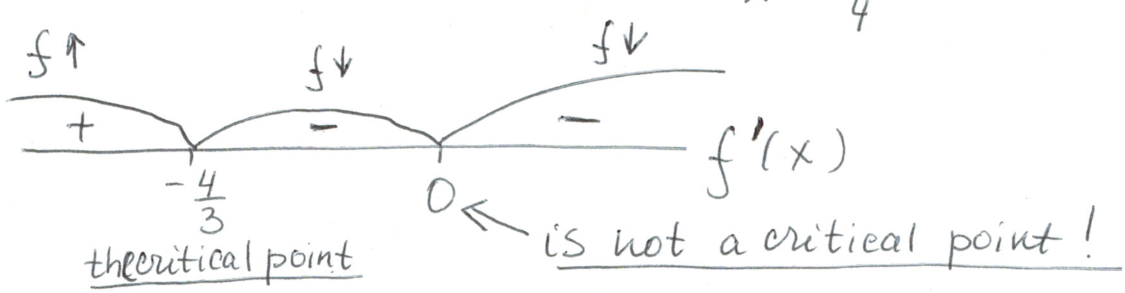
$$f'(x) = \frac{(2x+1)' \cdot 2x^3 - (2x^3)' \cdot (2x+1)}{4x^6} = \frac{4x^3 - 6x^2(2x+1)}{4x^6} =$$

$$= \frac{4x^3 - 12x^3 - 6x^2}{4x^6} = \frac{-8x^3 - 6x^2}{4x^6} = -\frac{(4x+3)}{2x^4}$$

$\boxed{\text{or}}$ $f'(x) = (x^{-2})' + \frac{1}{2}(x^{-3})' = -2x^{-3} - \frac{3}{2}x^{-4} = -\frac{2}{x^3} - \frac{3}{2x^4} = \frac{-4x-3}{2x^4}$

(d) The critical point(s) of f are $\boxed{f'(x) = 0 \Leftrightarrow 4x + 3 = 0}$

$$x = -\frac{3}{4}$$

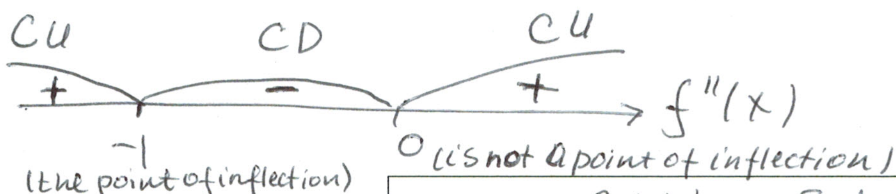


(e) The second derivative of f is $f'' = \boxed{\left(-2x^{-3} - \frac{3}{2}x^{-4}\right)' =}$

$$= 6x^{-4} - \frac{3}{2}(-4)x^{-5} = \frac{6}{x^4} + \frac{6}{x^5} = \frac{6x+6}{x^5}$$

$$f''\left(-\frac{3}{4}\right) = \frac{6\left(-\frac{3}{4}\right)+6}{\left(-\frac{3}{4}\right)^5} < 0 \implies x = -\frac{3}{4} \text{ is loc. maximum}$$

(f) The point(s) of inflection are $\boxed{f''(x) = 0 \iff 6x+6=0}$ or $\boxed{x=-1}$



(g) In the limit $\lim_{x\to 0^+} f(x) = \boxed{\lim_{x\to 0^+} \frac{2x+1}{2x^3} = \left[\frac{1}{+0}\right]} = +\infty$

(h) In the limit $\lim_{x\to 0^-} f(x) = \boxed{\lim_{x\to 0^-} \frac{2x+1}{2x^3} = \left[\frac{1}{-0}\right]} = -\infty$

$\boxed{\text{or}}$ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left[\frac{1}{x^2} + \frac{1}{2x^3}\right] = [-\infty - \infty] =$

$= \lim_{x\to 0^-} \frac{1}{x^2}\left(1 + \frac{1}{2x}\right) = +\infty \cdot \left[1 + \frac{1}{-0}\right] = +\infty \cdot (-\infty) = -\infty$

(i) In the limit $\lim_{x\to\infty} f(x) = \boxed{\lim_{x\to +\infty} \frac{2x+1}{2x^3} = \left[\frac{\infty}{\infty}\right]} =$

$$= \lim_{x\to\infty} \frac{2}{12x^2} = +0$$

(j) In the limit $\lim_{x\to-\infty} f(x) = \boxed{\lim_{x\to -\infty} \frac{2x+1}{2x^3} = +0}$

(k) The graph of f for $x \in [-2, 2]$ is