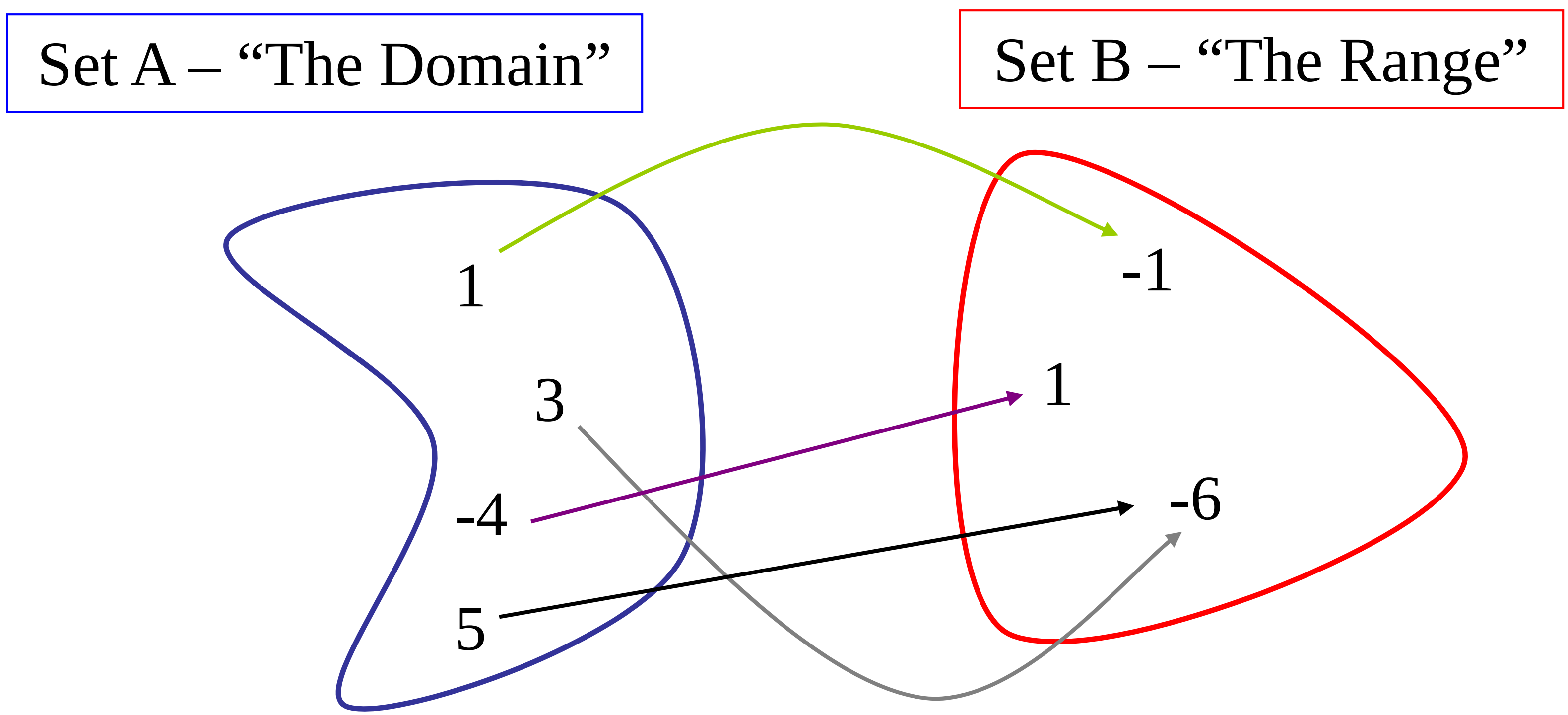
Chapter 2

Functions, Limits, and the Derivative

Functions

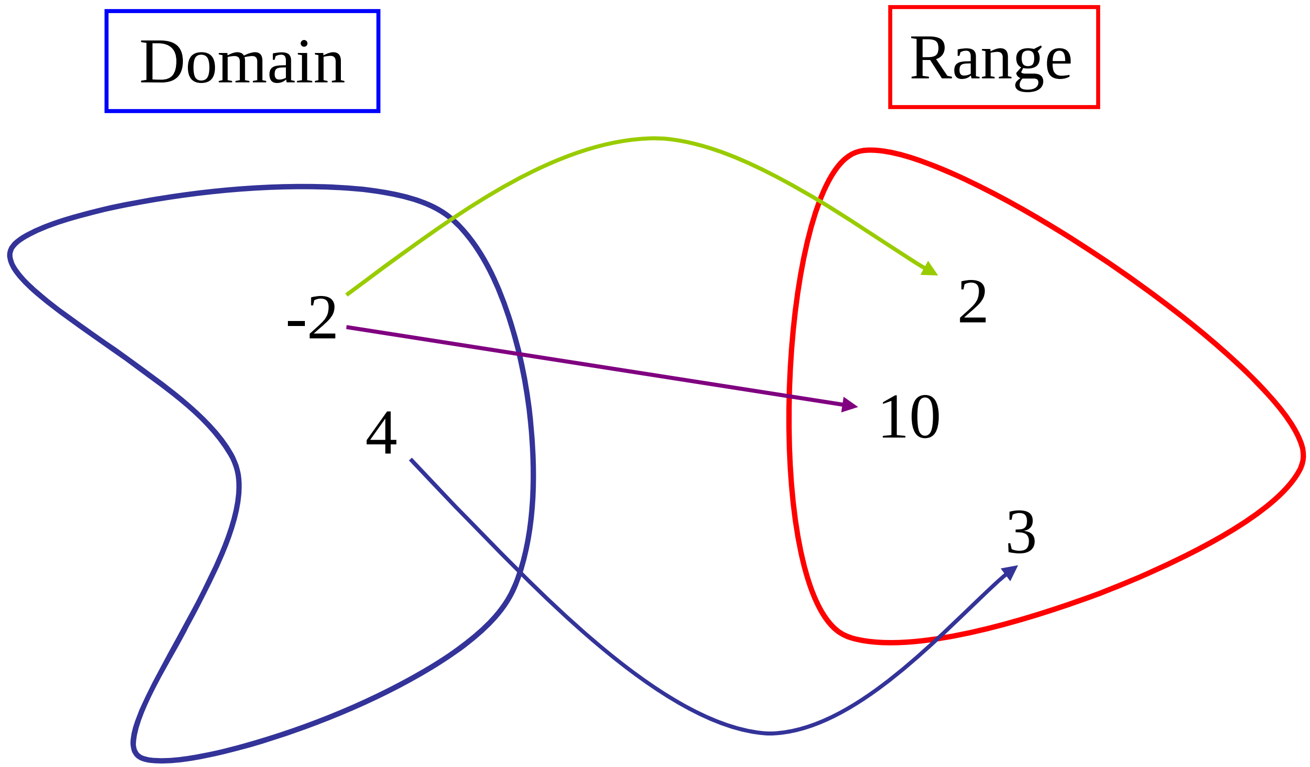
A **function** is a rule that assigns to each element in a set A one and only one element in a set B.



NOTE: We can use each element in the range more than once.

Functions

The following diagram does NOT represent a function.



Why not? There is one element in the domain (-2) assigned to two elements in the range (2 and 10).

Functions: Notation

Here is an example of the notation we use to represent a function:

$$y = 4x^2 + 5$$

The values of x correspond to the domain of the function. The values of y correspond to the range of the function.

We often replace y with $f(x)$. This is read as "f of x" or "f at the value x". This is NOT read as "f times x". Thus, the function example given above would be rewritten as:

$$f(x) = 4x^2 + 5$$

Functions: Notation

Each value of x in the domain of the function corresponds to one and only one value of y or $f(x)$ in the range of the function. We obtain these values in the range by simply "plugging in" the value of x into the function.

EXERCISE 1: Using the function on the previous page, what value in the range corresponds to

i) $x = 2$?

ii) $x = a + 4$, for some real number a?

Domain of a Function

The **domain** of a function f is the set of all real values x for which $f(x)$ is a real number.

EXERCISE 2: Determine the domain of the following functions:

i) $f(x) = 4x^2 + 5$

ii) $f(x) = \sqrt{3x+2}$

iii) $f(x) = \dfrac{8x}{5x-4}$

Range of a Function

The **range** of a function f is the set of all real values y such that there is a real value x for which $f(x) = y$.

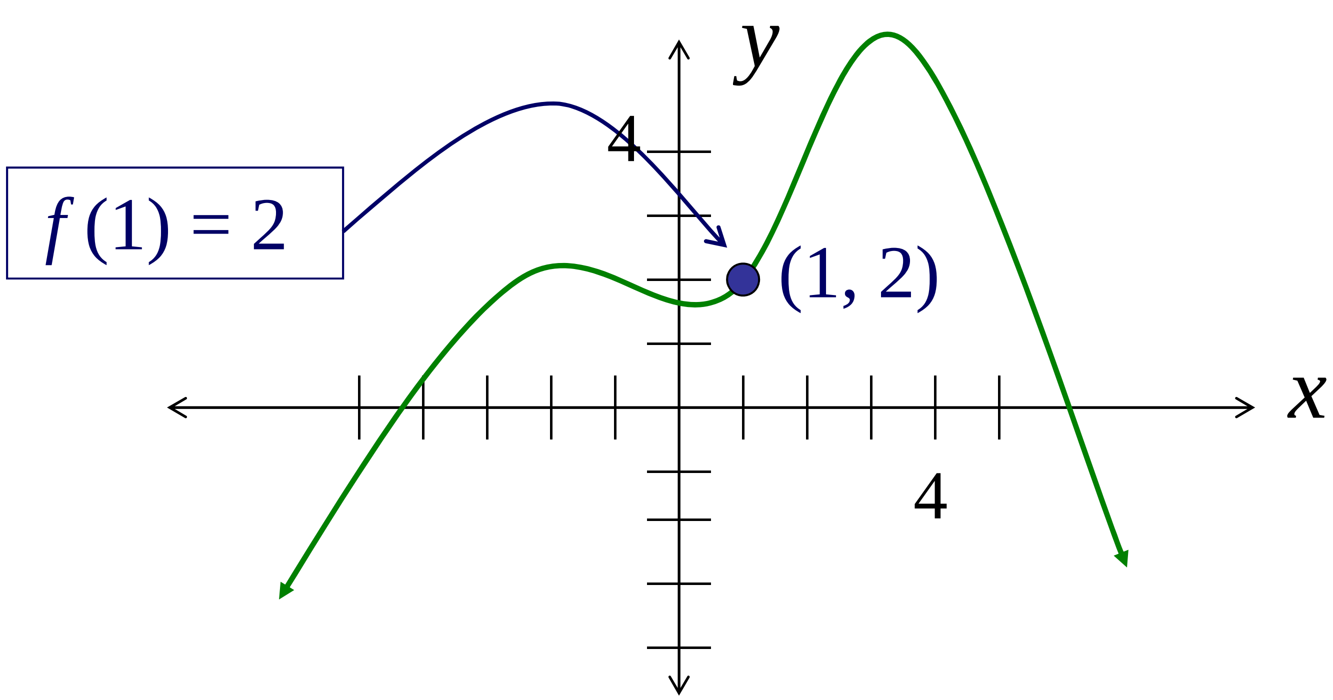
EXERCISE 3: Determine the range of the following functions:

i) $f(x) = 4x^2 + 5$

ii) $f(x) = \sqrt{3x+2}$

Graph of a Function

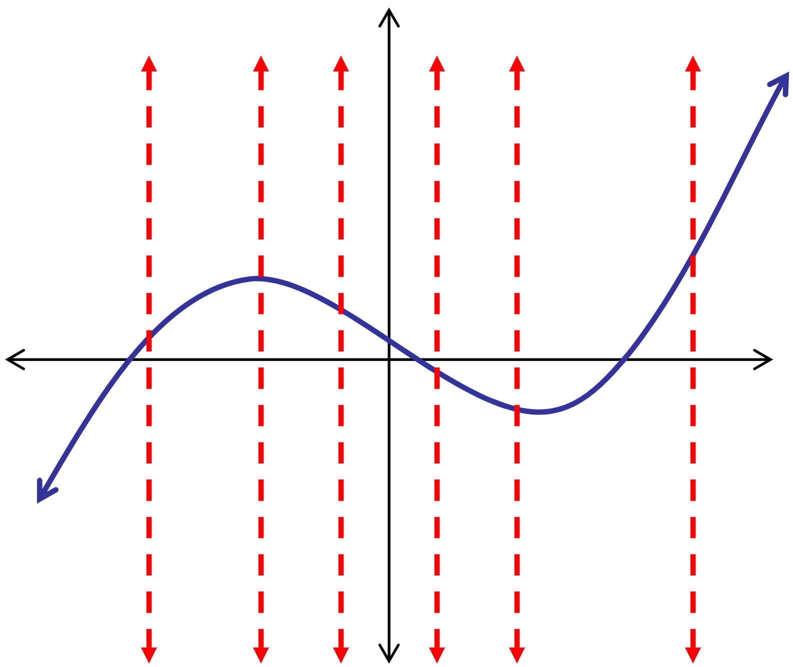
The **graph** of a function f is the set of all ordered pairs (x, y) in the xy–plane such that x is in the domain of f and $y = f(x)$.



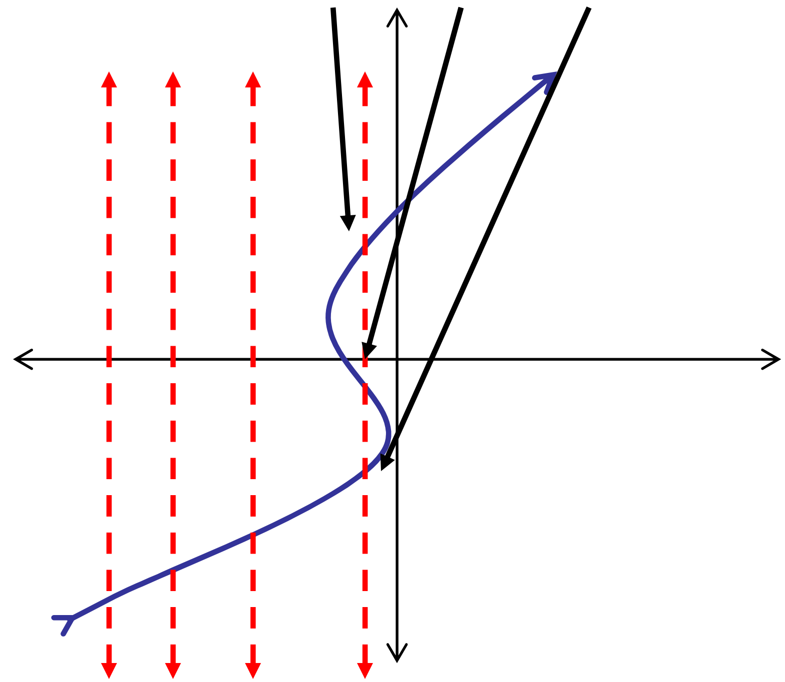
Graph of a Function

Vertical Line Test: The graph of a function can be crossed at most once by any vertical line.

Function



Not a Function



Composition of Functions

The composition of two functions f and g is the function $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$.

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ lies in the domain of f.

EXERCISE 4: Consider the functions $f(x) = 4x + 1$ and $g(x) = x^2 - 3x - 3$. Determine $(f \circ g)(1)$ and $(g \circ f)(-1)$. Also determine $(f \circ g)(x)$.

Types of Functions

A **polynomial function** of degree n is a function of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \qquad (a_0 \neq 0)$$

where $a_0, a_1, \ldots, a_n$ are real constants and n is a nonnegative integer.

EXAMPLES:

$f(x) = 4x^3 - 2x^2 + 10x - 5$ (cubic)

$g(x) = 2x^4 - 3x + 1$

$h(x) = x^2 - 3x + 1$ (quadratic)

Types of Functions

A **rational function** is a function of the form

$$R(x) = \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomial functions.

EXAMPLES:

$$F(x) = \frac{3x^3 + x^2 - x + 1}{x - 2}$$

$$G(x) = \frac{x^2 + 1}{x^2 - 1}$$

Motivating Example

Suppose you drive 200km in a period of 4 hours. What was your average speed?

$$\text{ave speed} = \frac{\text{distance covered}}{\text{elapsed time}} = \frac{\Delta y}{\Delta x} = \frac{200-0}{4-0} = 50\text{km/h}$$

Does this mean your speed is constantly 50km/h during the entire 4 hours? Of course not! If you read your speedometer at some random point during this trip, it might read 60km/h. This is your instantaneous speed.

Instantaneous speed at some given instant is equivalent to average speed as elapsed time approaches zero.

Motivating Example

EXERCISE 7: Suppose t represents time and $d(t) = t^2 + 10t$ represents your position (in metres) at time t (in seconds). Find

a) the average speed from $t = 1$ to $t = 3$.

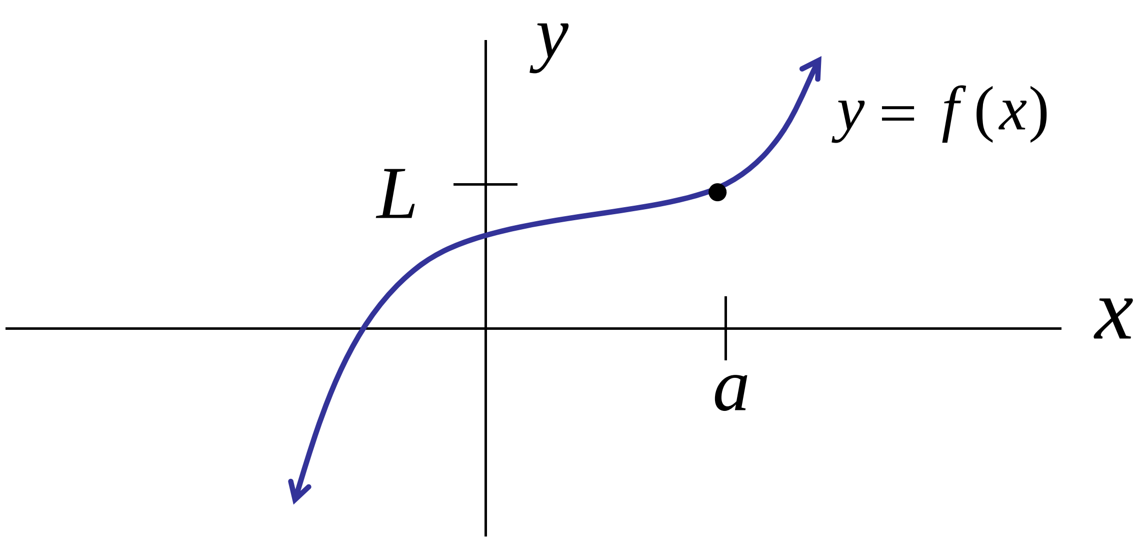
b) the instantaneous speed at $t = 1$.

Limit of a Function

The function f has the **limit** L as x approaches a, written

$$\lim_{x \to a} f(x) = L$$

if the value $f(x)$ can be made as close to the number L as we please by taking x sufficiently close to (but not equal to) a.



Limit of a Function

EXERCISE 8: Consider the following function:

$$f(x) = \begin{cases} -3x & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$$

Find $\lim_{x \to -2} f(x)$.

Theorem 1: Properties of Limits

Suppose

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M.$$

Then,

1) $\lim_{x\to a}[f(x)]^r = [\lim_{x\to a} f(x)]^r = L^r \quad (r \text{ real})$

2) $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cL \quad (c \text{ real})$

3) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$

4) $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)] = LM$

5) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \quad (M \neq 0)$

Indeterminate Forms

Suppose we are evaluating the limit of the quotient of two functions and the following happens:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}.$$

The resulting limit $0/0$ is known as an **indeterminate form**. This does not mean the limit does not exist, only that we haven't yet been able to determine the value of the limit.

We can evaluate this limit by replacing the given function with one that takes on the same values as the original function everywhere except at $x = a$.

Limits at Infinity

The function f has the **limit** L as x approaches infinity, written

$$\lim_{x\to\infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L taking x large enough.

The function f has the **limit** M as x approaches negative infinity, written

$$\lim_{x\to-\infty} f(x) = M$$

if $f(x)$ can be made arbitrarily close to M by taking x small enough.

Theorem 2: Limits at Infinity

For all $n > 0$,

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

provided that $\frac{1}{x^n}$ is defined.

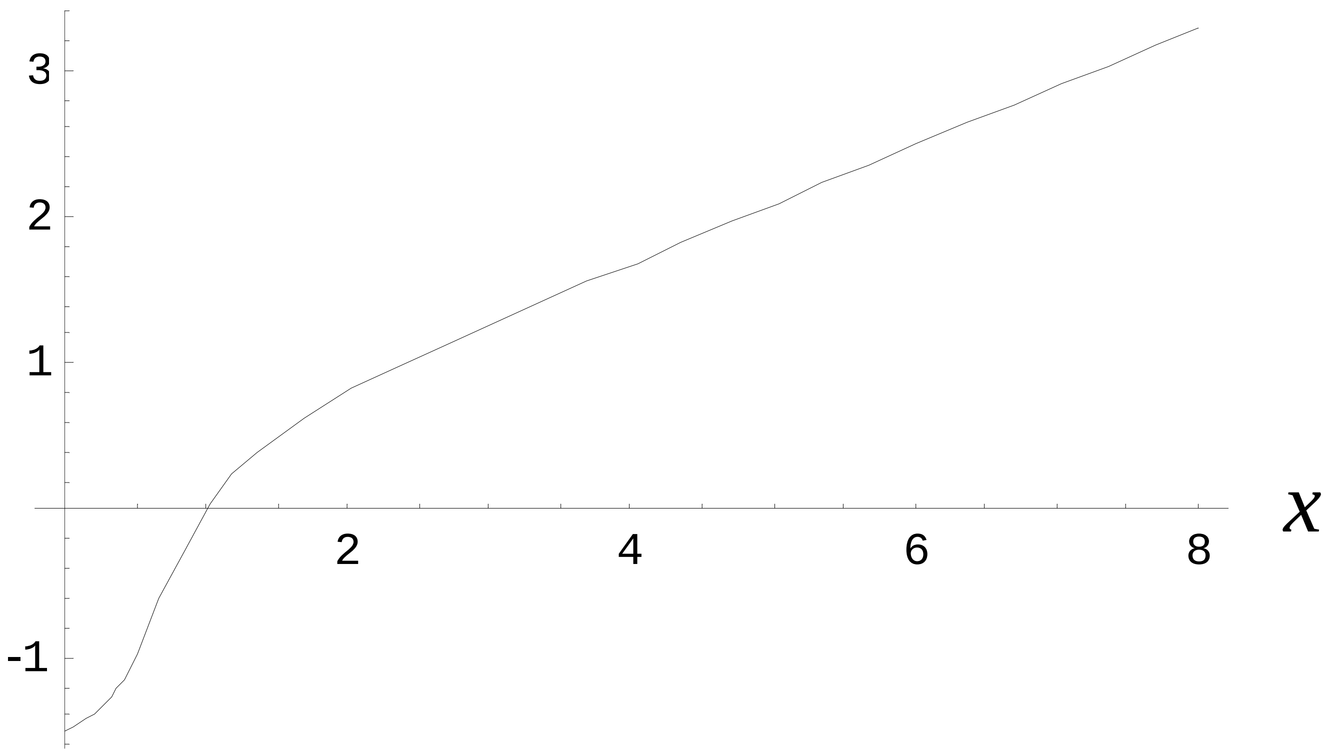
Limits at Infinity

When dealing with limits at infinity of rational functions, you should divide the numerator and denominator by the highest power of x in the denominator.

It will then become immediately clear if the limit of the function is a constant value, 0, or infinity.

Limits at Infinity

EXERCISE 14: Evaluate $\lim_{x \to \infty} \frac{2x^4 + x - 3}{5x^3 - x^2 + 2}$.



One Sided Limits of a Function

EXERCISE 15: Consider the function f defined by

$$f(x)=\begin{cases} x-1 & \text{if} \quad x<0 \\ x+1 & \text{if} \quad x\geq 0 \end{cases}.$$

Draw the graph of $f(x)$. Can we determine $\lim\limits_{x\to 0} f(x)$?

One Sided Limits of a Function

The function f has the **right-hand limit** L as x approaches a from the right, written

$$\lim_{x \to a^+} f(x) = L$$

if $f(x)$ can be made as close to L as we please by taking x sufficiently close to (but not equal to) a and to the right of a.

One Sided Limits of a Function

The function f has the **left-hand limit** M as x approaches a from the left, written

$$\lim_{x \to a^-} f(x) = M$$

if $f(x)$ can be made as close to M as we please by taking x sufficiently close to (but not equal to) a and to the left of a.

Theorem 3: Existence of the Limit

Let f be a function that is defined for all values of x close to $x = a$ with the possible exception of a itself. Then

$$\lim_{x \to a} f(x) = L \quad \text{if and only if}$$

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L.$$

In other words, the (two-sided) limit of a function exists if and only if the one-sided limits both exist and are equal to one another.

One Sided Limits of a Function

EXERCISE: Consider the following functions:

$$f(x) = \begin{cases} \sqrt{x} \text{ if } & x > 0 \\ -x \text{ if } & x \leq 0 \end{cases} \qquad g(x) = \begin{cases} -1 \text{ if } & x < 0 \\ 1 \text{ if } & x \geq 0 \end{cases}$$

a) Show that $\lim_{x \to 0} f(x)$ exists.

b) Show that $\lim_{x \to 0} g(x)$ does not exist.

Continuous Functions

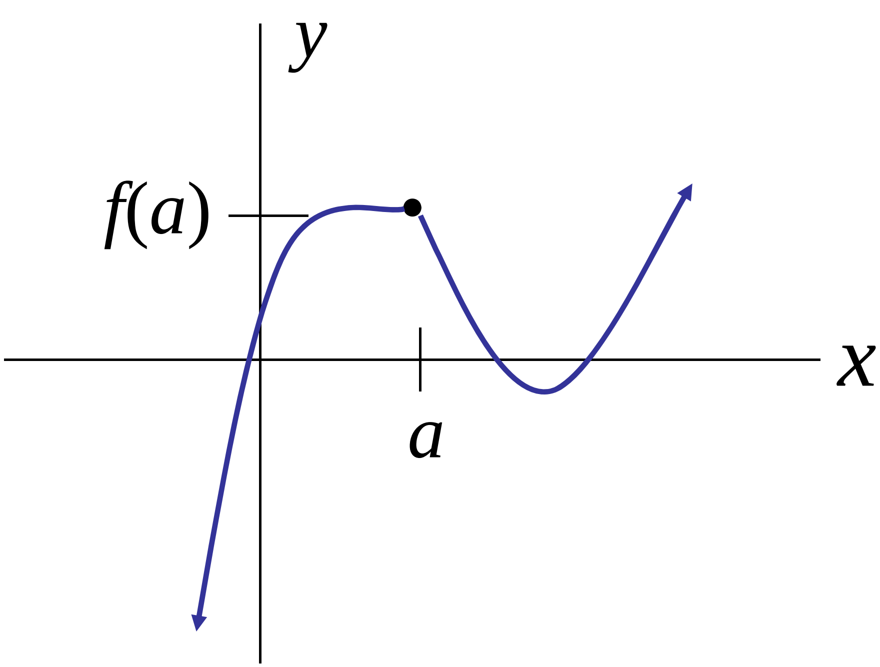
A function f is **continuous at the point** $x = a$ if the following three conditions are satisfied.

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Further, f is said to be **continuous on an interval** if f is continuous at every point in that interval.

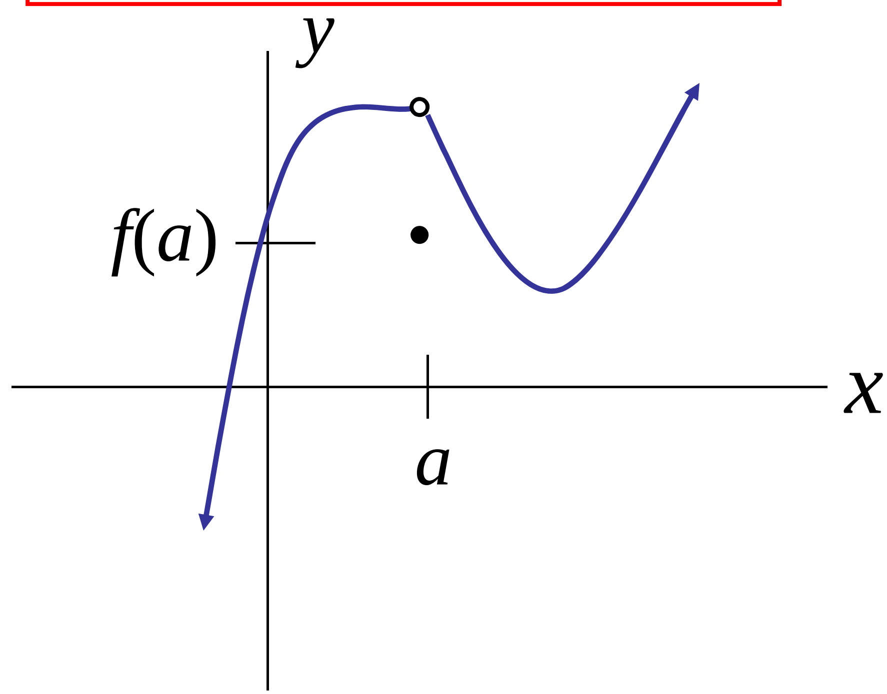
Continuous Functions

f is continuous at *a*



$$\lim_{x \to a} f(x) = f(a)$$

f is not continuous at *a*



$$\lim_{x \to a} f(x) \neq f(a)$$

Properties of Continuous Functions

1) The constant function $f(x) = c$ is continuous everywhere.

2) The identity function $f(x) = x$ is continuous everywhere.

If f and g are continuous at $x = a$, then

3) $[f(x)]^n$, where n is a real number, is continuous at $x = a$ whenever it is defined at the point.

4) $f \pm g$ is continuous at $x = a$.

5) fg is continuous at $x = a$.

6) f/g is continuous at $x = a$ provided $g(a) \neq 0$.

Continuity of Polynomial and Rational Fcns

1) A polynomial function $y = P(x)$ is continuous at every point x.

2) A rational function $R(x) = p(x)/q(x)$ is continuous at every point x where $q(x) \neq 0$.

EXERCISE: Find the values of x for which the following functions are continuous.

a) $f(x) = 3x^5 + 2x^3 - 5x + 1$

b) $g(x) = \dfrac{5x^4 - 2x + 3}{x^2 + 1}$

c) $h(x) = \dfrac{5x^4 - 2x + 3}{x^2 - 3x + 2}$

Rates of Change

The **average rate of change** of f over the interval $[x, x + h]$ is

$$\frac{f(x+h) - f(x)}{h}.$$

This can also be thought of as the **slope of the secant line** to the graph of f through the points $(x, f(x))$ and $(x + h, f(x + h))$.

Rates of Change

The **instantaneous rate of change** of f at x is

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

This can also be thought of as the **slope of the tangent line** to the graph of f at the point $(x, f(x))$.

The Derivative

The **derivative** of a function f with respect to x is the function f ' (read "f prime")

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of f ' is the set of all x where the limit exists.

Four-Step Process for Finding $f\,'(x)$

1. Compute $f(x + h)$.

2. Compute $f(x + h) - f(x)$.

3. Compute $\dfrac{f(x+h) - f(x)}{h}$.

4. Compute $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

The Derivative

EXERCISE: Consider the function $f(x) = 2x^2 - 1$.

Find $f'(x)$.

Differentiability and Continuity

If f is differentiable at $x = a$, then it is continuous at $x = a$. If it is not continuous at $x = a$, it is not differentiable at $x = a$.

NOTE: The reverse is not necessarily true. If f is continuous at $x = a$, it may or may not be differentiable at $x = a$.