PHY231H1

Final Exam Dec. 2012

What to study

Chapter 1 (1.11-1.15 – omit Bernoulli's equation)
Chapter 4 (4.1-4.3, 4.5-4.11 (omit 4.10.1), 4.12 (omit all equations after 4.61)
Chapter 5 (5.1-5.7)
Chapter 13 (13.1-13.8 (omit 13.7.1))

What to bring:

Calculator
Double-sided 8"×11" aid sheet prepared by you

What to do to get full marks:

Do all problems.

In order to get full marks, you have to provide an expression for the result and **only after that** you may substitute the numerical values.

Review problems:

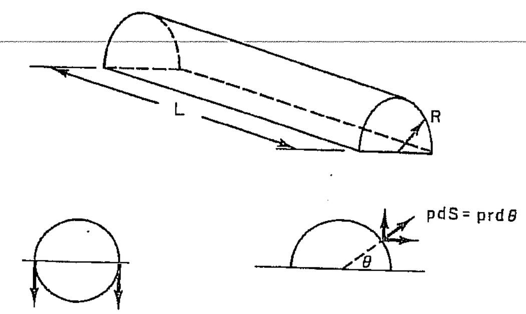
Chapter 1: 18, 19, 27, 41, 44, 47
Chapter 4: 11, 14, 17, 18, 21, 36, 44, 45, 49
Chapter 5: 13, 14, 17, 20, 26, 28
Chapter 13: 2, 9, 12, 15, 19, 24

Chapter 1

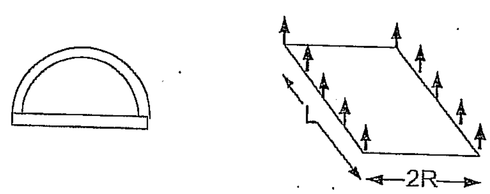
Problem 18 *The walls of a cylindrical pipe that has an excess pressure p inside are subject to a tension force per unit length T. (Consider only the force per unit length in the walls of the cylinder, not the force in any end caps of the pipe.) The force per unit length in the walls can be calculated by considering a different pipe made up of two parts as shown in the figure: a semicircular half-cylinder of radius R and length L attached to a flat plate of width 2R and length L. What is the force that the excess pressure exerts on the flat plate? Show that the tension force per unit length in the wall of the tube is $f = pR$. This is called the Law of Laplace. (Do not worry about any deformation.)*

See if you can obtain the same answer by direct integration of the horizontal and vertical components of the force due to the excess pressure.

Sometimes a patient will have an aneurysm in which a portion of an artery will balloon out and possibly rupture. Comment on this phenomenon in light of the R dependence of the force per unit length [Hademenos (1995)].



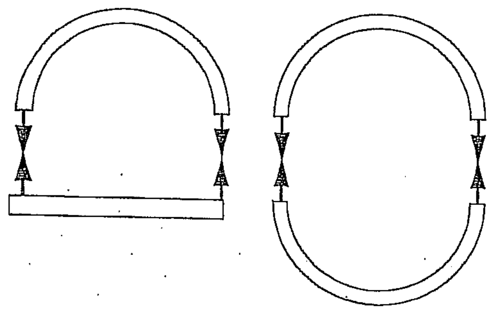
1.18 The total downward force on the flat plate is $F = pL2R$. If the plate is not accelerated, this must be balanced by a force exerted by the walls of the semi-circular segment. If the force per unit length is f, $2Lf = pL2R$, so $f = pR$.

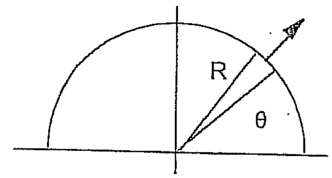


By the third law, the reaction to this is the force on the hemi-cylinder. This force is the same whether it is exerted by a flat plate or by another hemi-cylinder.

continued →

1.18 (continued)





We can get the same answer by direct integration of the force exerted on the hemi-cylinder by the gas inside. Consider the strip of length L and width $Rd\theta$. The force is pressure times area: $dF = pdS = pRL\, d\theta$.
The components are $dF_x = dF\cos\theta$ and $dF_y = dF\sin\theta$. The total force is obtained by integrating from $\theta = 0$ to $\theta = \pi$.

$$F_x = \int_0^\pi dF_x = pRL\int_0^\pi \cos\theta d\theta = pRL[\sin\theta]_0^\pi = pRL(0-0) = 0$$

$$F_y = \int_0^\pi dF_y = pRL\int_0^\pi \sin\theta d\theta = pRL[-\cos\theta]_0^\pi = pRL[-(-1)+(1)] = 2pRL$$

The force per unit length at the edge is $f = F_y/2L = pR$.

As the wall of an aortic aneurysm balloons out, R increases and the force per unit length also increases. Since the wall is already weakened to cause the original ballooning, the prognosis is not good.

Problem 19 *Find a relationship among the tension per unit length T across any element of the wall of a soap bubble, the excess pressure inside the bubble, Δp, and the radius of the bubble, R. (Hint: Use the same technique as for the previous problem.)*

1.19 In analogy with the previous problem, imagine a hemisphere of the soap bubble attached to a spherical flat plate. The force on the flat disk is $\Delta p\pi R^2$ and must be equal to the circumference times the tension per unit length, T: $\Delta p\pi R^2 = 2\pi RT \quad \Delta p = 2T/R$.

Problem 27 *Consider fluid flowing between two slabs as shown in Fig. 1.26. Since the work done by the external force on the system in time dt is* $dW = Fvdt$, *the rate of doing work is* $P = dW/dt = Fv$, *where* v *is the speed of the moving plate. Find the power dissipated per unit volume of the fluid in terms of the velocity gradient.*

1.27 If S is the area of the plate and L the spacing, the power is
$P = Fv = S\eta(dv/dy)v = S\eta v^2/L$
To get the power per unit volume, divide by the volume, SL:
$dP/dV = \eta v^2/L^2 = \eta(dv/dy)^2$.

Problem 41 *The velocity of the blood in the aorta is about 0.5 m* s^{-1}, *and the velocity of the blood in a capillary is about 0.001 m* s^{-1}. *We have only one aorta, with a diameter of 20 mm, but many capillaries in parallel, each with a diameter of 8* μm. *Estimate how many capillaries are typically open at any one time.*

1.41 The aorta is in series with the capillaries, so the volume current is the same in the aorta as in all N capillaries combined. The volume current is equal to the product of the speed and the area, So

$$\left(0.5\ \mathrm{m\,s^{-1}}\right)\pi(0.01\,\mathrm{m})^2 = N\left(0.001\ \mathrm{m\,s^{-1}}\right)\pi\left(4\times10^{-6}\,\mathrm{m}\right)^2 \qquad \text{or} \qquad N = 3 \text{ billion.}$$

Problem 44 *A sphere of radius* a *moving through a fluid with speed* v *is subject to a viscous drag* $F_{drag} = 6\pi\eta av$. *Make an argument similar to that in the text to show that the ratio of kinetic energy of a sphere of fluid of the same size moving at the same speed to the viscous work done to displace the sphere by its own diameter is* $N_R/18$.

1.44 The kinetic energy of a sphere of fluid is $E_k = m\,v^2/2 = (1/2)\rho\left(4\pi a^3/3\right)v^2$. The work done against the viscous force to drag the sphere a distance $2a$ is $W_{drag} = (6\pi\eta a\,v)(2a)$. The ratio is $\rho a\,v/18\eta = N_R/18$.

Problem 47 *Consider laminar viscous flow in the following situation, which models flow in the bronchi or a network of branching blood vessels. A vessel of radius R connects to N smaller vessels, each of radius xR.*

(a) What is the relationship between total cross-sectional area of the smaller vessels and that of the larger vessel if the pressure gradient is the same in both sets of vessels?

(b) How do the pressure gradients compare if the total cross-sectional area is the same in both sets of vessels? (Neither assumption is realistic.)

1.47 Let subscript 1 denote the large vessel; 2 the collection of small vessels. Then $i_1 = i_2$ so

$$\frac{\pi R_p^4}{8\eta}\left(\frac{\Delta p}{\Delta x}\right)_1 = \frac{N\pi R_p^4 x^4}{8\eta}\left(\frac{\Delta p}{\Delta x}\right)_2$$

(a) If $(\Delta p/\Delta x)_1 = (\Delta p/\Delta x)_2$ then $1 = Nx^4$ and $x^2 = 1/\sqrt{N}$

$S_2/S_1 = N\pi x^2 R_p^2/\pi R_p^2 = \sqrt{N}$

(b) If $S_2 = S_1$ then $Nx^2 = 1$, or $x^2 = 1/N$ so

$(\Delta p/\Delta x)_2/(\Delta p/\Delta x)_1 = 1/(Nx^4) = N^2/N = N$

Chapter 4

Problem 11 *Figure 4.12 shows that D for O_2 in water at 298 K is 1.2×10^{-9} m^2 s^{-1} and that the molecular radius of O_2 is 0.2 nm. The diffusion constant of a dilute gas (where the mean free path is larger than the molecular diameter) is $D = \lambda^2/2t_c$, where the collision time is given by Eq. 4.15.*

(a) Find a numeric value for the diffusion constant for O_2 in O_2 at 1 atm and 298 K and its ratio to D for O_2 in water. The molecular weight of oxygen is 32.

(b) Assuming that this equation for a dilute gas is valid in water, estimate the mean free path of an oxygen molecule in water.

4.11 **(a)** The mean free path in oxygen gas ($a_1 + a_2 = 2a$) is $\lambda = 1/(\pi 4a^2 N_A/V_M) = k_BT/(4\pi a^2 p) = (1.38 \times 10^{-23})(298)/(4\pi)(0.2 \times 10^{-9})^2(1.01 \times 10^5) = 8.1 \times 10^{-8}$ m $= 81$ nm, which is much larger than a. Combining this result with the equation for D and Eq. 4.15, we obtain

$$D = \frac{\lambda^2}{2t_c} = \frac{\lambda}{2}\left(\frac{3k_BT}{m}\right)^{1/2} = 1.95 \times 10^{-5}\ \mathrm{m^2 s^{-1}}.$$

This is much larger than for diffusion in water. The ratio is $1.95 \times 10^{-5}/1.2 \times 10^{-9} = 1.63 \times 10^4$.

(b) Using the equation blindly, we get $\lambda = 2D/(3k_BT/m)^{1/2}$ $= (2)(1.2\times10^{-9})/482 = 4.98\times10^{-12}$ m. This is much smaller than the molecular radius, so the equation is not necessarily valid.

Problem 14 *Write an equation for Fick's second law in three-dimensional Cartesian coordinates when the diffusion constant depends on position: $D = D(x, y, z)$.*

4.14 The fluence rate is $\mathbf{j}(x,y,z) = -D(x,y,z)[(\partial C/\partial x)\hat{\mathbf{x}} + (\partial C/\partial y)\hat{\mathbf{y}} + (\partial C/\partial z)\hat{\mathbf{z}}]$. The continuity equation is still

$$-\frac{\partial C}{\partial t} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}, \text{ so}$$

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial C}{\partial z}\right).$$

Since D depends on x y and z, it cannot be taken out in front of the partial derivative.

Problem 17 *A sheet of labeled water molecules starts at the origin in a one-dimensional problem and diffuses in the x direction.*

(a) Plot σ vs t for diffusion of water in water.

(b) Deduce a "velocity" versus time.

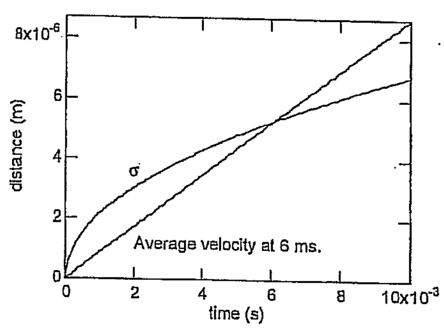
(c) How long does it take for the water to have a reasonable chance of traveling 1 μm? 10 μm? 100 μm? 1 mm? 1 cm? 10 cm?

4.17 At 310 K, D for water in water from Fig. 4.11 or 4.12 is 2.3×10^{-9} m^{2}s^{-1}. The standard deviation is $\sigma = \sqrt{2Dt} = \sqrt{4.6 \times 10^{-9} t}$.

t, s	σ, m	t, s	σ, m
0	0.	1000	2.1×10^{-3}
10^{-3}	2.1×10^{-6}	10^4	6.8×10^{-3}
2×10^{-3}	3.0×10^{-6}	10^5.	$21. \times 10^{-3}$
0.01	6.8×10^{-6}	10^6	$68. \times 10^{-3}$
0.1	$21. \times 10^{-6}$	10^8	$678. \times 10^{-3}$
1	$67. \times 10^{-6}$	10^{10}(=317 yr)	6.78
10	$214. \times 10^{-6}$		
100	$678. \times 10^{-6}$		

(b) The "velocity" depends on time interval. For about 6 ms, we can say

$$v = \frac{5.2 \times 10^{-6}\text{m}}{6 \times 10^{-3}\text{s}} = 8.7 \times 10^{-4}\text{ m s}^{-1}$$



(c) From the table

1 μm takes 1 ms or less

10 μm takes 0.1 s

100 μm takes 10 s

1 mm takes 1000 s

1 cm takes 10^5 s

10 cm takes 10^7 s

Problem 18 *In three dimensions the root-mean-square diffusion distance is $\sigma = \sqrt{6Dt}$, where t is the diffusion time. Consider the diffusion of oxygen from air to the blood in the lungs. The terminal air sacs in the lungs, the alveoli, have a radius of about 100 μm. The radius of a capillary is about 4 μm. Estimate the time for an oxygen molecule to diffuse from the center to the edge of an alveolus, and the time to diffuse from the edge to the center of a capillary. Which is greater? From the data in Table 1.4 estimate how long blood remains in a capillary. Is it long enough for diffusion of oxygen to occur? Assume the diffusion constant of oxygen in air is 2×10^{-5} m^2 s^{-1} and in water is 2×10^{-9} m^2 s^{-1}.*

4.18 The time for oxygen to diffuse through the air in the alveoli is $t = \frac{\sigma^2}{2D} = \frac{(100 \times 10^{-6}\,\text{m})^2}{2(2 \times 10^{-5}\,\text{m}^2\,\text{s}^{-1})} = 0.25\,\text{ms}$. The time for it to diffuse through the blood in the capillaries is $t = \frac{\sigma^2}{2D} = \frac{(4 \times 10^{-6}\,\text{m})^2}{2(2 \times 10^{-9}\,\text{m}^2\,\text{s}^{-1})} = 4\,\text{ms}$. So, most of the time is spent diffusing through the blood, not the air. Capillaries are about 1 mm long, and the speed of the blood in them is about 10^{-3} m s^{-1}. So, blood spends about 1 s in the capillary, which should provide more than enough time for oxygen diffusion.

Problem 21 *A substance has diffusion constant D, and its concentration is distributed in space according to $C(x,t) = A(t)\sin(2\pi x/L)$, where L is the wavelength and $A(t)$ is the amplitude of the distribution. Use the one-dimensional diffusion equation, Eq. 4.26, to show that the concentration decays exponentially with time, $A(t) \propto e^{-t/\tau}$. Determine an expression for the time constant τ in terms of L and D. Which decays faster: a long-wavelength (diffuse) distribution, or a short-wavelength (localized) distribution? This result can be used with the Fourier methods developed in Chapter 11 to derive very general solutions to the diffusion equation.*

4.21 Substitute $C(x,t) = A(t)\sin(2\pi x/L)$ into the diffusion equation $\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$, to get $\frac{dA}{dt} = -D\left(\frac{2\pi}{L}\right)^2 A$. The solution to this differential equation is $e^{-\frac{t}{\tau}}$, where $\tau = \frac{L^2}{4\pi^2 D}$. Localized concentration distributions have small L and therefore small τ; they decay quickly. Diffuse concentration distributions have large L and therefore large τ; they decay slowly.

Problem 36 *The distance L that oxygen can diffuse in the steady state is approximately* $L = \sqrt{CD/Q}$, *where C is the oxygen concentration, D is the diffusion constant, and Q is the rate per unit volume that oxygen is used for metabolism.*

(a) Show that L has dimensions of length.

(b) The diffusion of oxygen in air is about 10,000 times larger than the diffusion of oxygen in water [Denny (1993)]. By how much will the diffusion distance L change if oxygen diffuses through air instead of water, all other things being equal?

Insects deliver oxygen to their flight muscles by diffusion down air-filled tubes instead of by blood vessels, thereby taking advantage of the large diffusion constant of oxygen in air [Weiss-Fogh (1964)].

4.36 **(a)** C has units of molecules m^{-3}.
D has units of m^2 s^{-1}.
Q has units of molecules m^{-3} s^{-1}.

Therefore, L has units of $\sqrt{\dfrac{\left(\dfrac{\text{molecules}}{\text{m}^3}\right)\left(\dfrac{\text{m}^2}{\text{s}}\right)}{\dfrac{\text{molecules}}{\text{m}^3\ \text{s}}}} = \sqrt{\text{m}^2} = \text{m}$.

(b) If D grows by 10,000, then L grows by $\sqrt{10{,}000} = 100$. So, oxygen will diffuse one hundred times as far in air as in water.

Problem 44 *We can write the diffusion constant, D, and the thermal speed* v_{rms} *in terms of the step size,* λ, *and the collision time,* t_c *as*

$$D = \frac{\lambda^2}{2t_c},$$

$$v_{rms} = \frac{\lambda}{t_c}.$$

Solve for λ *and* t_c *in terms of D and* v_{rms}.

4.44 $\lambda = \dfrac{2D}{v_{\text{rms}}}$ and $t_c = \dfrac{2D}{v_{\text{rms}}^2}$.

Problem 45 *Using the definitions in Problem 44, write the diffusion constant in terms of λ and v_{rms}. By how much do you expect the diffusion constant for "heavy water" (water in which the two hydrogen atoms are deuterium, ^2H) to differ from the diffusion constant for water? Assume the mean free path is independent of mass.*

4.45 $D=\frac{\lambda v_{rms}}{2}$. We assume that λ is independent of mass, but v_{rms} varies as the inverse square root of mass (Eq. 4.12). The mass of normal water is 2(1)+1(16)=18, whereas the mass of heavy water is 2(2)+1(16)=20. The mass increases by (20-18)/18 or 11%, so the diffusion constant decreases by 3.3%.

Problem 49 *The text considered a one-dimensional random-walk problem. Suppose that in two dimensions the walk can occur with equal probability along $+x$, $+y$, $-x$, or $-y$. The total number of steps is $N = N_x + N_y$, where the number of steps along each axis is not always equal to $N/2$.*

(a) What is the probability that N_x of the N steps are parallel to the x axis?

(b) What is the probability that the net displacement along the x axis is $m_x\lambda$?

(c) Show that the probability of a particle being at $(m_x\lambda, m_y\lambda)$ after N steps is

$$P'(m_x, m_y) = \sum_{N_x}\left(\frac{N!}{N_x!\,(N-N_x)!}\right)\left(\frac{1}{2}\right)^N P(m_x, N_x)\,P(m_y, N-N_x),$$

where $P(m, N)$ on the right-hand side of this equation is given by Eq. 4.76.

(d) The factor $N!/N_x!(N-N_x)!$ is proportional to a binomial probability. What probability? Where does this factor peak when N is large?

(e) Using the above result, show that $P'(m_x, m_y) = P(m_x, N/2)\,P(m_y, N/2)$.

(f) Write a Gaussian approximation for two-dimensional diffusion.

4.49 $N=N_x+N_y$

(a) The probability of moving parallel to the x axis is 1/2. The probability that N_x of the N moves are parallel to the x axis is

$$P(N_x;N)=\frac{N!}{N_x!(N-N_x)!}\left(\frac{1}{2}\right)^{N_x}\left(\frac{1}{2}\right)^{N-N_x}$$

(b) There are N_x steps along the x axis, which may be positive or negative with equal probability. The number to the right is n, and the number to the left is n'. $n+n'=N_x$. The net number to the right is $m=n-n'$. Then $n=(N_x+m)/2$ and $n'=(N_x-m)/2$.

$$P(m;N_x)=\frac{N_x!}{\left(\frac{N_x+m}{2}\right)!\left(\frac{N_x-m}{2}\right)!}\left(\frac{1}{2}\right)^{N_x}$$

continued →

4.49 continued

(c) There are many ways to end up with a given pair of values m_x and m_y. The probabilities for each independent way add. For example, if there are ten steps, there could be 0, 1, 2, . . . of these parallel to x axis. For each case we must calculate probabilities for m_x and m_y and add them:

$$P(m_x, m_y) = \sum_{N_x} \frac{N!}{N_x!(N-N_x)!}(^1/_2)^N P(m_x; N_x) P(m_y; N - N_x)$$

where $N_y = N - N_x$.

(d) $N!/N_x!(N-N_x)!$ is proportional to binomial probability $P(N_x; N)$ when $p = q = 1/2$. It peaks at $N_x = N/2$.

(e)

$$P(m_x, m_y) = \sum_{N_x}\left[\frac{N!(^1/_2)^N}{N_x!(N-N_x)!}\right] P(m_x; N_x) P(M_y; N_y)$$

Since the factor in square brackets peaks near $N_x = N_y = N/2$, assume that the P's, which are slowly varying functions of N_x and N_y, can be moved outside the sum. Then

$P(m_x, m_y) = P(m_x; N/2) P(M_y; N/2) \sum_{N_x}\left[\frac{N!(^1/_2)^N}{N_x!(N-N_x)!}\right]$. The sum is unity, so

$P(m_x, m_y) = P(m_x; N/2) P(m_y; N/2)$.

(f) We need a Gaussian approximation for $P(m_x, N/2)$ and $P(m_y, N/2)$. (Both have the same functional form.) We follow the arguments of Sec. 4.12, noting that only every other value of m_x or m_y occurs. Therefore $P(x,t) = \left(N\pi\lambda^2\right)^{-1/2}\exp(-m_x^2/N)$. It is still true that $N = t/t_c$. In addition, $x = m_x\lambda$, $y = m_y\lambda$. Therefore

$P(x,t) = \left(t_c/\pi t\lambda^2\right)^{1/2}\exp(-x^2 t_c/\lambda^2 t)$. When this is combined with a similar expression for y, we find

$$P(x,y,t) = \left(\frac{t_c}{\pi\lambda^2 t}\right)^{2/2}\exp[-(x^2+y^2)t_c/\lambda^2 t].$$

The form of the equation is the same as in one dimension if we define $4D = \lambda^2/t_c$.

$$P(x,y,t) = \left(\frac{1}{4\pi Dt}\right)\exp[-(x^2+y^2)/4Dt].$$

Chapter 5

Problem 13 *Solute is carried through a pipe by solvent drag. The radius of the pipe is b. The average flow along the pipe is $\bar{j}_v$ (independent of r because it has been averaged over r). Assume that within the pipe the concentration of solute is independent of radius and can be written as $C(z)$. The solute is carried along purely by solvent drag. Solute concentration outside the pipe is zero. Solute diffuses through the wall of the pipe, which has solute permeability ωRT. In terms of $\bar{j}_v$, b, and ωRT, obtain a differential equation for $C(z)$ and show that C decays exponentially along the pipe. Find the decay constant.*

5.13 Consider an element of the pipe between z and $(z + dz)$. The number of particles into the element at z is $j_v C(z)\pi b^2$. The number out at $(z + dz)$ is $j_v C(z + dz)\pi b^2$. The number out through the walls of the element is $\omega RTC(z)2\pi b dz$. To conserve particles, $j_v[C(z) - C(z+dz)]\pi b^2 = \omega RTC(z)2\pi b dz$

$$j_v \pi b^2 \frac{dC}{dz} = -(\omega RT 2\pi b)C$$

$$\frac{dC}{dz} = -\left(\frac{2\omega RT}{j_v b}\right)C$$

which is the equation for exponential decay along the pipe, with decay constant $\frac{2\omega RT}{j_v b}$.

Problem 14 *A kidney machine has a membrane permeability $\omega RT = 0.5 \times 10^{-3}$ cm s^{-1}. If the membrane area is 1 m^2, the volume of body fluid is 40 l, and the volume of dialysant is effectively infinite, what is the time constant? How long will it take to reduce the BUN (blood urea nitrogen) concentration from 120 mg per 100 ml to 20 mg per 100 ml?*

5.14 The time constant is given by Eq. 5.22:

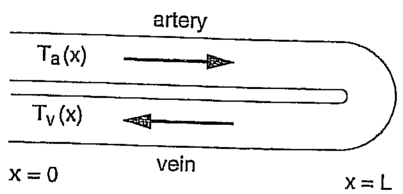
$$\tau = \frac{V}{S\omega RT} = \frac{40 \text{ liter}}{(1 \text{ m}^2)(0.5\times 10^{-3} \text{cm s}^{-1})}\left(\frac{10^{-3}\text{m}^3}{\text{liter}}\right)\left(100\frac{\text{cm}}{\text{m}}\right) = 8\times 10^3 \text{ sec or 2.22 hours.}$$

To reduce the urea concentration from 120 to 20:

$$\frac{20}{120} = e^{-t/\tau} \qquad \ln(20/120) = -1.79 = -\frac{t}{8\times 10^3}$$

$$t = 1.43\times 10^4 s = 239 \text{ min} = 4 \text{ hours.}$$

Problem 17 *The countercurrent model applies to the transport of heat as well as particles, with temperature taking the place of concentration. Consider a countercurrent heat exchanger, which represents the arrangement of blood vessels in the flipper of a whale [Schmidt-Nielsen (1972)].*



The temperatures of the arterial and venous blood are governed by equations similar to Eqs. 5.27:

$$T_a = c_1 + (c_2 - c_1)ax,$$
$$T_v = c_2 + (c_2 - c_1)ax.$$

Assume that the arterial blood at $x = 0$ is at the warm temperature of the whale's body, T_w. The arterial blood at $x = L$ enters the capillaries at temperature $T_a(L)$ and is cooled to the temperature of the surrounding ocean water, T_c, by the time it enters the vein at $x = L$.

(a) Determine c_1 and c_2 in terms of T_w, T_c, a, and L.

(b) Plot $T_a(x)$ and $T_v(x)$ for $T_w = 37\,^\circ\mathrm{C}$, $T_c = 7\,^\circ\mathrm{C}$, $a = 1\,\mathrm{mm}^{-1}$, and $L = 3\,\mathrm{mm}$.

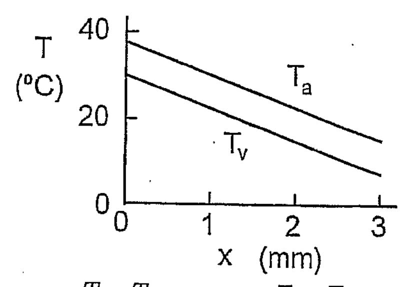
(c) The loss of heat from the body to the surroundings is proportional to $\Delta T = T_a(L) - T_c$. Find an expression for ΔT. What does ΔT reduce to if $aL \ll 1$? if $aL \gg 1$? Interpret these results physically. To minimize heat loss to the ocean should aL be large or small?

(d) The energy the body must supply to heat the returning venous blood is proportional to $\Delta T' = T_w - T_v(0)$. Find an expression for $\Delta T'$.

5.17 **(a)** At $x = 0$, $T_a = T_w$: $c_1 = T_w$.

At $x = L$, $T_v = T_c$: $c_2 = \dfrac{T_c + T_w aL}{1 + aL}$.

(b)



(c) $\Delta T = T_w + \dfrac{T_c - T_w}{1 + aL} aL - T_c = \dfrac{T_w - T_c}{1 + aL}$.

If $aL \ll 1$, then $\Delta T = T_w - T_c$. In this case, L is much less than the length constant ($1/a$), so very little change in temperature occurs along the vessels.

If $aL \gg 1$, then $\Delta T = \dfrac{T_w - T_c}{aL}$. In this case, L is much greater than the length constant, so the temperature becomes almost equal in the artery and vein, and the temperature difference, ΔT, goes to zero. To minimize heat loss to the ocean, aL should be large.

(d) $\Delta T' = \Delta T = \dfrac{T_w - T_c}{aL}$.

Problem 20 *Obtain expressions for J_s when $\lambda = 0$ and $\lambda \to \infty$.*

5.20 The condition $\lambda \to 0$ means $\lambda/\Delta Z \to 0$. Write

$$J_s = (1-\sigma)\overline{C}_s J_v\left(1+\frac{\omega RT\,\Delta C_s}{(1-\sigma)\overline{C}_s J_v}\right) = (1-\sigma)\overline{C}_s J_v\left(1+\frac{\lambda}{\Delta Z}\frac{\Delta C_s}{\overline{C}_s}\right).$$

The second term in brackets vanishes as $\lambda \to 0$.

The other condition is $\lambda/\Delta Z \to \infty$. Factor the equation as

$$J_s = \omega RT\,\Delta C_s\left(1+\frac{(1-\sigma)\overline{C}_s J_v}{\omega RT\,\Delta C_s}\right) = \omega RT\,\Delta C_s\left(1+\frac{\Delta Z}{\lambda}\frac{\overline{C}_s}{\Delta C_s}\right)$$

In the limit $\lambda/\Delta Z \to \infty$, the second term is negligible.

Problem 26 *Consider the following cases for transport of water through a membrane.*

(a) Water flows by bulk flow through the membrane with $\Delta p = 0$. There is an impermeant solute ($\sigma = 1$) on the right with concentration C_{big} and zero concentration on the left. Find the particle fluence rate of water in terms of L_p.

(b) There is no volume flow through the membrane ($J_v = 0$). Some of the water molecules on the left are tagged with radioactive hydrogen (tritium). The concentration of tagged water molecules is C_s on the left and 0 on the right. Find the particle fluence rate of tagged water in terms of L_p and ωRT.

(c) There is volume flow, as in case (a), and there are also tagged water molecules on the left. Find the particle fluence rate of tagged water in terms of L_p and ωRT.

(d) Restate the answers in terms of the parameters of a collection of n pores per unit area of radius R_p and length ΔZ.

(e) Estimate the value of x for part (c) if $R_p = 10^{-}$ m and $c_{big} = c_s = 0.1$ mol l^{-1}.

5.26 (a) The volume flow is $J_v = L_p k_B T C_{\text{big}}$. The water particle fluence rate is $J_w = C_w J_v = C_w L_p k_B T C_{\text{big}}$, where, in terms of Avogadro's number and the molar volume of water, $C_w = N_A / \overline{V}_w$.

(b) If $J_v = 0$, $J_s = \omega RTC_s$.

(c) With both volume flow and diffusion and $C_s' = 0$, $J_s = J_v(1-\sigma)\overline{C}_s + \omega RTC_s$. To evaluate this, we need the average concentration, which is $\overline{C}_s = C_s/2 + G(x)C_s$. The parameter x is

$x = L_p k_B T C_{\text{big}} / \omega RT$. Alternatively, once we know x we could use $J_s = \omega RTC_s\left(\frac{xe^x}{e^x - 1}\right)$.

continued →

5.26 continued

(d) For part a, $J_w = \dfrac{n\pi R_p^4 k_B T C_w C_{\text{big}}}{8\eta\Delta Z}$. For part b, $J_s = \dfrac{n\pi R_p^2 D C_s}{\Delta Z}$, where D is the diffusion constant of water in water. For part c,

$$x = \frac{L_p k_B T C_{\text{big}}}{\omega RT} = \frac{n\pi R_p^4 k_B T C_{\text{big}} \Delta Z}{8\eta \Delta Z\, n\pi R_p^2 D} = \frac{R_p^2 k_B T C_{\text{big}}}{8\eta D}.$$

(e) From Fig. 4.12 for water in water, $D = 3.3 \times 10^{-9}$ m^2 s^{-1}.

$$C_{\text{big}} = \frac{0.1 \text{ mol}}{1} \frac{10^3}{\text{m}^3} \frac{6\times 10^{23} \text{ molecule}}{\text{mol}} = 6\times 10^{25} \text{ molecule m}^{-3}$$

$$x = \frac{(10^{-8})^2(1.38\times 10^{-23})(300)(6\times 10^{25})}{(8)(10^{-3})(3.3\times 10^{-9})} = 0.94.$$

Problem 28 *Consider the case of water permeability shown in Fig. 5.1(c). Water and solute molecules move through the membrane in the same way. They "dissolve" from solution into the membrane. Assume that the concentration of water molecules just inside the membrane is proportional to the pressure just outside:* $C = \alpha p$. *The membrane has thickness* ΔZ *and the diffusion constant for water in the membrane material is* D. *Under steady-state conditions, derive an expression for* L_p.

5.28 In the steady state $\partial C/\partial t = 0$ implies that $\partial j_s/\partial z = 0$ which means that $j_s = -D\partial C/\partial z$ is independent of z. Therefore $\partial C/\partial z \to \Delta C/\Delta Z \to \alpha\Delta p/\Delta Z$. The fluence rate of water molecules is $J_s = D(\Delta C/\Delta Z) = (D\alpha/\Delta Z)\Delta p$. But $L_p = J_v/\Delta p$. To convert from J_s to J_v multiply by the volume of one molecule, the molar volume divided by N_A: $L_p = \overline{V}_w D\alpha / N_A \Delta Z$. (We could have defined the solubility to be in mole per Pa. Then C would have been in mole m^{-3} and the factor N_A would not appear.)

Chapter 13

Problem 2 *Show that the pressure p satisfies the wave equation. Hint: Use Eqs. 13.13 and 13.14. Differentiate to obtain $\partial^2 p/\partial x^2$ and $\partial^2 p/\partial t^2$. Also use the fact that when multiple partial derivatives are taken, the order of differentiation can be interchanged (Appendix N).*

13.2 The time derivative of Eq. 13.13 gives $\frac{\partial^2 v}{\partial t \partial x} = -\kappa \frac{\partial^2 p}{\partial t^2}$.

The space derivative of Eq. 13.14 gives $\frac{\partial^2 v}{\partial x \partial t} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x^2}$.

The order of differentiation does not matter, so we find that $\frac{\partial^2 p}{\partial t^2} = \frac{1}{\kappa \rho_0}\frac{\partial^2 p}{\partial x^2}$, which is the wave equation.

Problem 9 *Derive the relationships between p_0, ξ_0 and v_0 (Eqs. 13.22 and 13.23), where p_0, ξ_0 and v_0 are the amplitudes of a sinusoidally varying plane wave.*

13.9 $p = p_0 \sin(kx - \omega t)$, $\xi = \xi_0 \cos(kx - \omega t)$, $v = v_0 \sin(kx - \omega t)$.

From Eq. 13.8, $p = -\frac{1}{\kappa}\frac{\partial \xi}{\partial x}$, we get $p_0 \sin(kx - \omega t) = -\frac{1}{\kappa}(-k\xi_0 \sin(kx - \omega t))$, or $\xi_0 = \frac{\kappa}{k} p_0$ (equivalent to Eq. 13.22)

From Eq. 13.12, $v = \frac{\partial \xi}{\partial t}$, we get $v_0 \sin(kx - \omega t) = \omega \xi_0 \sin(kx - \omega t)$, or $\xi_0 = \frac{1}{\omega} v_0$.

If we eliminate ξ_0 from these two equations and recall that $\omega = kc$, $c = \frac{1}{\sqrt{\rho_0 \kappa}}$, and $Z = \sqrt{\frac{\rho_0}{\kappa}}$, we find $v_0 = \frac{p_0}{Z}$ (Eq. 13.23).

Problem 12 *The threshold for audible sound is 10^{-12} W m^{-2}. Use Eq. 13.29 to convert this to the amplitude of the pressure oscillation in air, using $Z_{air} = 400$ Pa s m^{-1}. Compare this to 10^5 Pa (atmospheric pressure), and to 5×10^{-6} Pa (which is on the order of the amplitude of random pressure variations in the air due to thermal motion). Are the pressure oscillations small? Perform the same analysis for the threshold for pain, $I = 10^{-4}$ W m^{-2}.*

13.12 $p = \sqrt{2IZ} = \sqrt{2(10^{-12})(400)} = 28 \times 10^{-6}$ Pa. This pressure is much smaller that atmospheric pressure, and is only about five times the pressure in air due to thermal motion.

$p = \sqrt{2(10^{-4})(400)} = 0.28$ Pa. This pressure is still much smaller than atmospheric pressure, but is now much larger than pressure caused by thermal motion.

Problem 15 *Use the results of Problems 13 and 14 to show that* $R + T = 1$.

13.15 From Prob. 13.13, $R = \frac{I_r}{I_i} = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2$. From Prob. 13.14, $T = \frac{I_t}{I_i} = \frac{4Z_1Z_2}{(Z_2 + Z_1)^2}$.

$$R + T = \frac{(Z_2 - Z_1)^2}{(Z_2 + Z_1)^2} + \frac{4Z_1Z_2}{(Z_2 + Z_1)^2} = \frac{Z_2^{\,2} + Z_1^{\,2} - 2Z_1Z_2 + 4Z_1Z_2}{(Z_2 + Z_1)^2} = \frac{Z_2^{\,2} + Z_1^{\,2} + 2Z_1Z_2}{(Z_2 + Z_1)^2} = \frac{(Z_2 + Z_1)^2}{(Z_2 + Z_1)^2} = 1.$$

Problem 19 *The ear can just hear sound at about* 1,000 Hz *at a level that corresponds to a pressure change of* 2×10^{-5} Pa. *Atmospheric pressure is* 10^5 Pa. *Since atmospheric pressure is due to collisions of molecules with the eardrum, there are pressure fluctuations because of fluctuations in the number of collisions in time* Δt. *We can expect that* $\Delta p/p$ *is about* $1/(\text{number of collisions})^{1/2}$. *Suppose that the eardrum has area* S *and that when detecting a signal at* 1,000 Hz *it averages over a time interval of* 0.5 ms. *The number of collisions per unit area per unit time is given by* $nv/4$, *where* n *is the number of air molecules per unit volume and* v *is an average velocity of* $482\,\mathrm{m\,s^{-1}}$. *The radius of the eardrum is 4.5 mm. Find* $\Delta p/p$.

see over →

13.19 The number of molecules per unit volume, n, of air can be estimated from the ideal gas law, $pV = Nk_BT$.

$$n = \frac{N}{V} = \frac{p}{k_BT} = \frac{\left(10^5\,\text{Pa}\right)}{\left(1.4\times10^{-23}\,\text{J/K}\right)\left(310\text{K}\right)} = 2.3\times10^{25}\text{molecules/m}^3.$$

The number of collisions is then $\frac{nv}{4}\pi r^2\,\Delta t$: $\Delta p/p$ is one over the square root of the number of collision

$$\frac{\Delta p}{p} = \frac{1}{\sqrt{\frac{nv}{4}\pi r^2\,\Delta t}} = \frac{1}{\sqrt{\frac{\left(2.3\times10^{25}\text{molecules/m}^3\right)\left(482\,\text{m/s}\right)}{4}\pi\left(4.5\times10^{-3}\text{m}\right)^2\left(0.5\times10^{-3}\text{s}\right)}}$$

$$\frac{\Delta p}{p} = 10^{-10}.$$

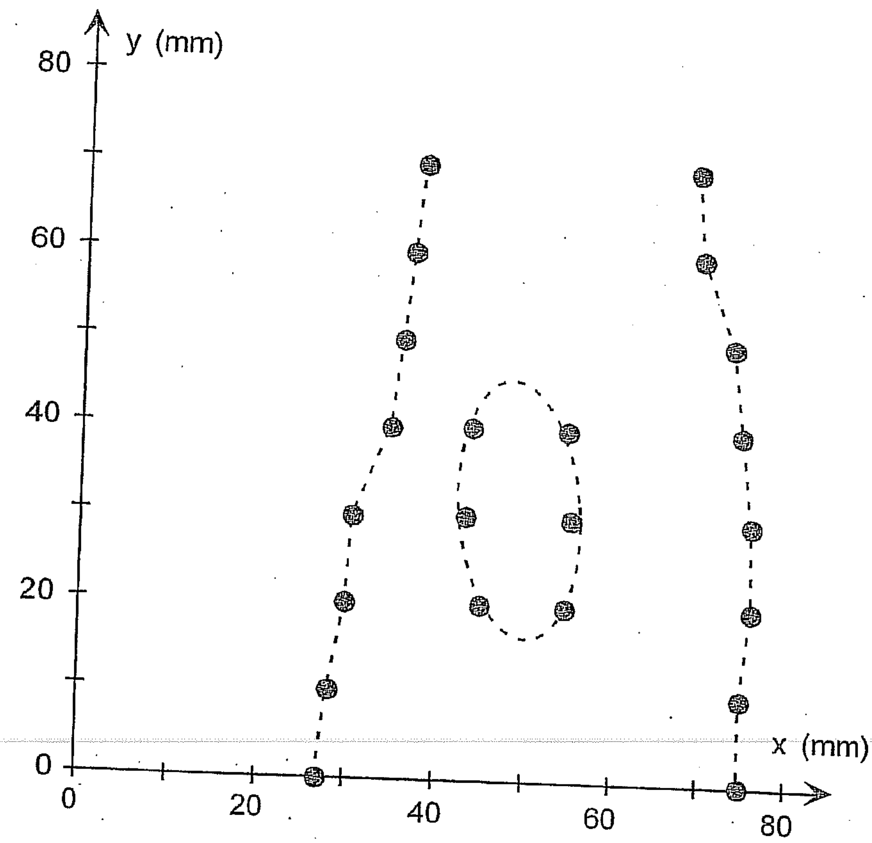
Problem 24 *Suppose you emit an ultrasound pulse in the x direction from a source at each of eight different positions y. Each pulse receives a series of echoes, as shown in the table below (Echo times are in μs.):*

y (mm):	0	10	20	30	40	50	60	70
Echo 1	35	37	39	40	45	47	48	49
Echo 2	97	98	58	56	57	96	91	90
Echo 3			71	73	71			
Echo 4			99	99	98			

Draw an x-y coordinate system ($x = 0$ is location of the source) and put a bright spot corresponding to each echo. Assume $c = 1,540\ m\ s^{-1}$ in each tissue, and ignore attenuation. You have just created a two-dimensional ultrasound image.

13.24 Below is the two-dimensional image. The bright spots correspond to each echo. The dashed curve is an eyeball guess as to the geometry of each boundary.

see over →



y (mm)
80
60
40
20
0
0
20
40
60
80
x (mm)