University of Ottawa
MAT 1332B Midterm Exam

Feb. 13, 2008. Duration: 80 minutes. Instructor: Frithjof Lutscher

Solutions

Question 1. [3 points] Consider the two functions

$$f(x) = \frac{1}{1+2x}, \qquad g(x) = 1 - x.$$

1. (1 point) Show that the functions intersect at the two points $x_1 = 0$ and $x_2 = 1/2$.
2. (2 points) Find the area enclosed by the two functions between the two points of intersection.

Solution: For the intersection points, we solve $f(x) = g(x)$, which gives the quadratic equation

$$(1-x)(1-2x) = 1, \qquad \text{or} \qquad 2x^2 - x = 0 \qquad \text{or} \qquad x(2x-1) = 0.$$

Hence, the intersections occur at $x_1 = 0$ and $x_2 = 1/2$. Since $f(1/4) = 2/3 < 3/4 = g(1/4)$, we know that $g(x) > f(x)$ for $x_1 < x < x_2$. Therefore the area enclosed by the two curves is given by

$$\int_{x_1}^{x_2} (g(x) - f(x))dx = \int_0^{1/2} \left(1 - x - \frac{1}{1+2x}\right) dx = \left(x - \frac{x^2}{2} - \frac{1}{2}\ln(|1+2x|)\right)\Big|_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{2}\ln(2) - 0 = \frac{3}{8} - \frac{1}{2}\ln(2).$$

Question 2. [5 points] Suppose that the mass of a worm grows according to the equation

$$\frac{dW}{dt} = 4e^{-t/2}, \qquad W(0) = 10.$$

The units are in milligrams.

1. (2 point) How much is the mass gain between $t = 0$ and $t = 2$?
2. (2 points) Is the mass gain until $t = \infty$ finite or infinite?
3. (1 point) Will the mass of the worm ever reach 20 mg?

Solution: The mass gain is given by the definite integral

$$\int_0^2 \frac{dW}{dt} dt = \int_0^2 4e^{-t/2} dt = -8(e^{-t/2})|_0^2 = 8(1 - e^{-1}).$$

Hence, the mass gain is approximate 5.057 mg.

The mass gain until infinity is

$$\int_0^\infty \frac{dW}{dt} dt = \lim_{T\to\infty} \int_0^T 4e^{-t/2} dt = \lim_{T\to\infty} -8(e^{-t/2})|_0^T = \lim_{T\to\infty} 8(1 - e^{-T/2}) = 8.$$

In particular the weight gain until infinity is finite. Also, since the initial weight is 10mg and the weight gain until infinity is only 8mg, the mass of the worm will never reach 20mg.

Question 3. [4 points] Find the indefinite integral

$$\int \frac{7x-1}{x^2+4x-5} dx.$$

Solution: The denominator can be factored as $x^2 + 4x - 5 = (x-1)(x+5)$. Then we set up partial fractions as

$$\frac{A}{x-1} + \frac{B}{x+5} = \frac{(A+B)x + 5A - B}{x^2+4x-5}.$$

Comparison with the numerator of the integrand gives the two conditions

$$A + B = 7, \qquad 5A - B = -1.$$

The solution is $A = 1, B = 6$. Then we integrate as follows

$$\int \frac{7x-1}{x^2+4x-5} dx = \int \frac{1}{x-1} + \frac{6}{x+5} dx = \ln(|x-1|) + 6\ln(|x+5|) + C.$$

Question 4. [2 points] Does the following improper integral converge or diverge? If it converges, give its value.

$$\int_0^\infty \frac{1}{(2+x)^2} dx.$$

Solution: By definition and using the substitution $u = 2 + x$, we have

$$\int_0^\infty \frac{1}{(2+x)^2} dx = \lim_{T\to\infty} \int_0^T \frac{1}{(2+x)^2} dx = \lim_{T\to\infty} \int_2^T \frac{1}{u^2} du = \lim_{T\to\infty} (-u^{-1})|_2^T = \lim_{T\to\infty} (-1/T + 1/2) = 1/2.$$

In particular, the integral converges.

Question 5. [7 points] In the following, we describe the migration of birds between a small island and the mainland. We denote by I and M the density of birds on the island and the mainland, respectively. The units are birds per square kilometer. We assume that birds leave

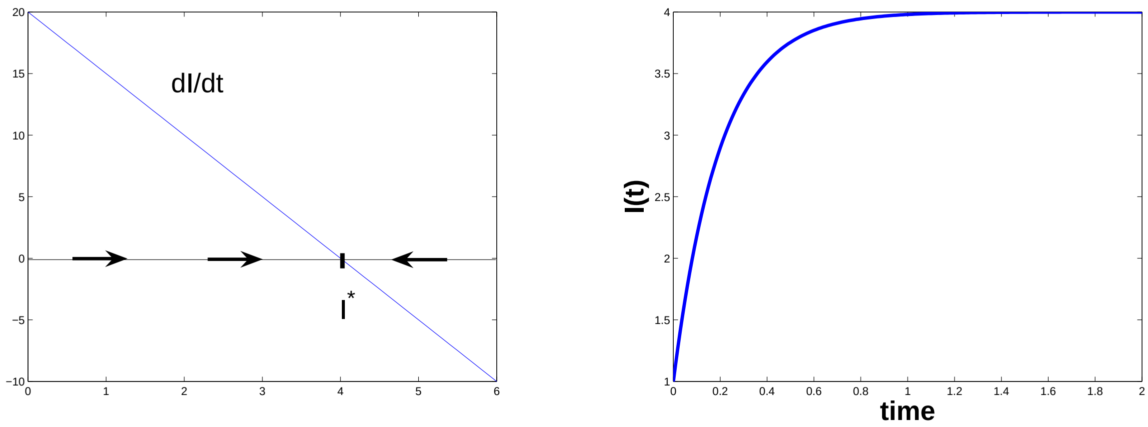

Figure 1: Plots for Question 5

the island at a rate k and that birds from the mainland arrive on the island at a rate a. Then the equation for the bird density on the island reads

$$\frac{dI}{dt} = aM - kI.$$

We assume that the mainland is so large that the density of birds there is constant, i.e., the few individuals that leave for the island are negligible compared to the total number. Hence, M is constant in time. Choose the following parameter values: $a = 1, k = 5, M = 20$.

1. (2 points) Find the steady state I^*. Determine its stability, using the derivative test. What is the density of birds on the island in the long run?

2. (2 points) Draw the phase line diagram and sketch the solution curve $I(t)$, starting at $I(0) = 1$.

3. (3 points) Find an explicit solution to the equation with initial value $I(0) = 1$. [Hint: from the phase line diagram, you know that the solution satisfies $I(t) < 4$ for all $t > 0$.]

Solution: The steady state is given when $dI/dt = 0$, which is when $I = aM/k = 4$. Differentiating the right hand side with respect to I gives simply $-k < 0$. Therefore, by the derivative test, the steady state is stable. Hence, in the long run, the density of birds on the island will approach 4 per square kilometer.

For the phase line diagram and the solution curve, see Figure 1.

For the explicit solution, we separate variables and integrate.

$$\int \frac{dI}{aM - kI} = \int dt.$$

This leads to

$$-\frac{1}{k}\ln(|aM - kI|) = t + C.$$

Since we start with $I(0) = 1$ we have $I(t) < aM/k = 4$, for all times, and hence we drop the absolute value signs and take exponentials on both sides.

$$aM - kI = e^{-kt}e^{-kC}.$$

Solving for $I(t)$ gives

$$I(t) = aM/k - \frac{e^{-kC}}{k}e^{-kt}.$$

The initial value $I(0) = 1$ gives $e^{-kC} = aM - k$, or the final solution

$$I(t) = aM/k - \frac{aM - k}{k}e^{-kt} = 4 - 3e^{-5t}.$$

Question 6. [3 points] Find the population $P(t)$ if the growth rate is the oscillating function $3 - \cos(t)$ with $P(0) = 1000$, i.e., solve the separable differential equation

$$\frac{dP}{dt} = (3 - \cos(t))P, \qquad P(0) = 1000.$$

Solution: Separation of variables gives

$$\int \frac{dP}{P} = \int (3 - \cos(t))dt.$$

Integration gives

$$\ln(|P|) = 3t - \sin(t) + C.$$

Solving for positive P makes

$$P(t) = e^{3t-\sin(t)}e^{C}.$$

From the initial value, we get $e^C = 1000$.

Question 7. [6 points] Consider the autonomous differential equation

$$\frac{dP}{dt} = P^2 - 6P + 8.$$

Do not solve this equation explicitly!

1. (1 point) Find the steady states P_1^*, P_2^*.
2. (2 point) Find the stability of P_1^*, P_2^*, using the derivative test.
3. (1 point) Draw the phase line diagram.
4. (2 points) Without solving the equation explicitly sketch the graph of $P(t)$ starting at $P(0) = 3.5$. Indicate inflections points if there are any.

Solution: The right hand side is $f(P) = (P - 2)(P - 4)$. The steady states are therefore $P_1^* = 2$ and $P_2^* = 4$. The derivative of f is $f' = 2P - 6$, so that $f'(2) = -2 < 0$ and $f'(4) = 2 > 0$. Hence, P_1^* is stable and P_2^* is unstable. The inflection point is at $P = 3$ where $f'(3) = 0$. For phase-line diagram and solution, see Figure 2.

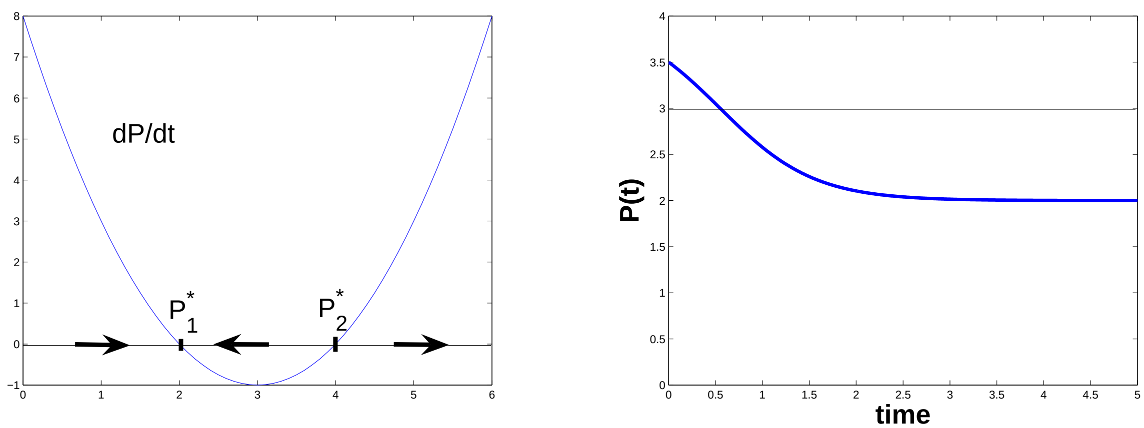

Figure 2: Plots for Question 7