MAT1332 Spring/Summer 2009

Midterm Exam 1 Solutions.

Problem 1 (6 points)

Suppose the quantity v of a radioactive isotope decays according to the law

$$\frac{dv}{dt} = -kv \tag{1}$$

with $k > 0$ a constant, v measured in moles and t measured in years. Assume that at time $t = 0$ there are 20 moles of the isotope and at $t = 5$ there 19.25 moles remaining.

(a) Solve the differential equation (1), with the initial condition given above, using separation of variables.

(b) Find the value of k using your solution to (a).

(c) Determine the half-life of the isotope using your solutions to (a) and (b).

(d) How much of the isotope remains after 20 years?

For (a) we separate variables $\frac{1}{v}dv = -kdt$ and integrate to obtain $\int \frac{1}{v}dv = -k\int dt$. The left-hand side of this equation is $\ln|v| + C_1$ and the right-hand side is $-kt + C_2$, where C_1 and C_2 are constants of integration. Thus, combining the constants of integration $C_0 := C_2 - C_1$, we have $\ln|v| = -kt + C_0$. Exponentiating gives

$$|v| \;=\; e^{\ln|v|} \;=\; e^{-kt+C_0} \;=\; Ce^{-kt}$$

where $C := e^{C_0}$. We are only considering the case where v takes non-negative values and therefore, in the range of values we consider, $v = Ce^{-kt}$. The initial condition $v(0) = 20$ gives $20 = Ce^0 = C$ and we have, in the range of values we consider, $v = 20e^{-kt}$.

For (b), note that we are given that $v(5) = 19.25$. As such, we have, using our solution from (a), $19.25 = 20e^{-5k}$. Dividing gives

$$\frac{77}{80} = e^{-5k}.$$

Taking the logarithm of both sides then yields $-5k = \ln\left(\frac{77}{80}\right)$ and

$$k \;=\; -\frac{1}{5}\ln\left(\frac{77}{80}\right) \;\approx\; 0.0076.$$

For (c), we must fine the time t such that $v(t) = \frac{1}{2}v(0)$. Using our solutions to (a) and (b) we see that this equation holds if and only if $20e^{-0.0076t} = 10$. I.e., if and only if $e^{-0.0076t} = \frac{1}{2}$. Taking the logarithm of both sides yields $-0.0076t = \ln(\frac{1}{2})$ and

$$t \;=\; -\frac{\ln(\frac{1}{2})}{0.0076} \;=\; \frac{\ln(2)}{0.0076} \;\approx\; 91.2036.$$

Thus the half-life of the isotope is approximately 91.2036 years.

For (d), we calculate $v(20) = 20e^{(-0.0076)(20)} \approx 17.1798$. Thus, after 20 years approximately 17.1798 moles remain.

Problem 2 (3 points)

Does the following improper integral converge or diverge? If it converges, give its value.

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx.$$

First let us find the indefinite integral $\int \frac{\ln(x)}{x^2}dx$. There are several ways to find the integral. One is to use integration by parts with $f(x) := -\frac{1}{x}, g(x) := \ln(x)$. Instead I will show you a way to do it using the fact that $\ln(x)$ is the inverse of e^x (this kind of substitution is useful in general if you need to integrate a function containing $\ln(-)$ terms). Substituting $x := e^u$, so that $u = \ln(x)$ and $\frac{dx}{du} = e^u$, we have

$$\int \frac{\ln(x)}{x^2}dx = \int \frac{u}{e^{2u}}e^u du = \int \frac{u}{e^u}du.$$

Integration by parts then yields

$$\int \frac{u}{e^u}du = -\frac{u}{e^u} + \int \frac{1}{e^u}du = -\frac{u}{e^u} - \frac{1}{e^u} + C.$$

Substituting back $u = \ln(x)$ gives

$$\int \frac{\ln(x)}{x^2}dx = -\frac{\ln(x)}{x} - \frac{1}{x} + C.$$

Writing the improper integral as a limit of definite integrals, we obtain

$$\int_1^\infty \frac{\ln(x)}{x^2}dx = \lim_{N\to\infty}\int_1^N \frac{\ln(x)}{x^2}dx = \lim_{N\to\infty}\big(-\frac{\ln(x)}{x} - \frac{1}{x}\big)\Big|_1^N = \lim_{N\to\infty}\big(-\frac{\ln(N)}{N} - \frac{1}{N} + 1\big).$$

Now, this is equal to $\lim_{N\to\infty}(-\frac{\ln(N)}{N}) - \lim_{N\to\infty}\frac{1}{N} + 1$. But we have $\lim_{N\to\infty}\frac{1}{N} = 0$ and also $\lim_{N\to\infty}(-\frac{\ln(N)}{N}) = 0$ (the latter can be seen using l'Hôpital's rule). Thus,

$$\int_1^\infty \frac{\ln(x)}{x^2}dx = 1.$$

Problem 3 (4 points)

Calculate the indefinite integral

$$\int \frac{x^3+2x^2-18x+2}{x^2+x-12}dx.$$

First we apply long division to obtain

$$\int \frac{x^3+2x^2-18x+2}{x^2+x-12}dx = \int xdx + \int dx + \int \frac{-7x+14}{x^2+x-12}dx$$

Now, $x^2+x-12=(x+4)(x-3)$. As such, in order to integrate $\int \frac{-7x+14}{x^2+x-12}dx$ we would like to find constants A and B such that

$$\frac{-7x+14}{x^2+x-12} = \frac{A}{x+4} + \frac{B}{x-3},$$

for any x. In particular, we should have $-7x+14=(x-3)A+(x+4)B$ for any x. Taking $x=3$ gives $B=-1$. Taking $x=-4$ on the other hand gives $A=-6$. So

$$\int \frac{-7x+14}{x^2x-12}dx \ = \ -6\int \frac{1}{x+4}dx - \int \frac{1}{x-3}dx \ = \ -6\ln|x+4| - \ln|x-3| + C.$$

Thus,

$$\int \frac{x^3+2x^2-18x-12}{x^2+x-12}dx \ = \ \frac{x^2}{2} + x - 6\ln|x+4| - \ln|x-3| + C.$$

Problem 4 (5 points)

Suppose the amount of chemical produced obeys

$$\frac{dP}{dt} = 8e^{-2.5t}$$

with initial condition $P(0) = 12$. Here t is measured in minutes and P in moles.

(a) How much chemical is produced between $t = 0$ and $t = 5$?

(b) Is the production until $t = \infty$ finite or infinite?

(c) Will the total amount of chemical produced ever be 17 moles?

For (a), we integrate

$$\int_0^5 8e^{-2.5t}\,dt \;=\; -3.2e^{-2.5t}\Big|_0^5 \;=\; -3.2e^{-12.5} + 3.2 \;\approx\; 3.2000.$$

Thus, between $t = 0$ and $t = 5$ there are approximately 3.2000 moles of chemical produced.

For (b), we must determine the value of the improper integral $\int_0^\infty 8e^{-2.5t}dt$. This is calculated as follows:

$$\int_0^\infty 8e^{-2.5t}dt \;=\; \lim_{N\to\infty}\int_0^N 8e^{-2.5t}dt \;=\; \lim_{N\to\infty}(-3.2e^{-2.5N} + 3.2) \;=\; 3.2,$$

since $\lim_{N\to\infty} e^{-2.5N} = 0$. Thus, the production until $t = \infty$ is finite.

For (c), since there are only 3.2 moles produced between $t = 0$ and $t = \infty$, there are never 17 moles produced. (In fact, there will not even ever be a total of 17 moles of the chemical since there are only 15.2 moles of the chemical after $t = \infty$).

Problem 5 (5 points)

Consider the differential equation

$$\frac{dV}{dt} = V^3 + 3V^2 - 18V.$$

Without solving this equation explicitly:

(a) What are the equilibria of the equation?

(b) Which of these equilibria are stable and which are unstable?

(c) If V is a solution satisfying the initial condition $V(0) = 3$, then what are its inflection points (if any)?

For (a), note that $V^3+3V^2-18V = V(V^2+3V-18) = V(V-3)(V+6)$ and therefore the equilibria are $V = 0$, $V = 3$ and $V = -6$.

For (b), we use the derivative test. If $F(V) = V^3 + 3V^2 - 18V$, then $F'(V) = 3V^2 + 6V - 18$. As such, $F'(0) = -18$ and the equilibrium $V = 0$ is stable. On the other hand $F'(3) = 27$ and $F'(-6) = 54$. Thus, both equilibria $V = 3$ and $V = -6$ are unstable.

For (c), since $V = 3$ is one of the equilibria such a solution has no inflection points.

Problem 6 (3 points)

Define functions f and g by

$$f(x) := 3x^2 \qquad \text{and} \qquad g(x) := -3x+6.$$

(a) Show that f and g intersect at points $x=-2$ and $x=1$.

(b) Calculate the area between the f and g in the interval $-2 \le x \le 1$.

(a) can be solved in a number of ways. E.g., you can note that $f(x)-g(x) = 3(x-1)(x+2)$, or you could just compute the values at -2 and 1: $f(-2) = 3(-2)^2 = 3 \times 4 = 12$ and $g(-2) = 6+6 = 12$, and $f(1) = 3$ and $g(1) = -3+6 = 3$.

For (b), we first must determine which curve lies above the other in this region. Since $f'(x) = 6x$ and $g'(x) = -3$ we have that, at $x=-2$, $f(x)$ is decreasing at a greater rate than $g(x)$. Thus, the area A between f and g in this interval is given by

$$A = \int_{-2}^{1} g(x) - f(x) dx = (-\frac{3}{2}x^2 + 6x - x^3)\Big|_{-2}^{1} = (-\frac{3}{2} + 6 - 1) - (-6 - 12 + 8) = \frac{27}{2} = 13.5.$$

Problem 7 (Bonus: 6 points)

Consider a tank of water with the shape of the surface of revolution obtained by rotating the curve $f(x) := \frac{1}{3}\sqrt{x}$ about the x-axis between 1 and 5. The narrow portion of the tank is closer to the ground and we measure the height of the tank in meters so that the tank is 4 meters tall. Let A_x denote the area of the (circular) cross-section of the tank at height x for $1 \leq x \leq 5$.

Suppose furthermore that there is a circular hole in the base of the tank with diameter $\frac{1}{5}$ meters. Denote by $H(t)$ the height (in meters) of the body of water in the tank (measured from the base of the tank to the water-line) at time t (with t measured in minutes). Then the rate of change of H is given by the differential equation

$$\frac{dH}{dt} = -\frac{ka\sqrt{H}}{A_H},$$

where $k > 0$ is a constant and a is the area of the hole (in square meters).

(a) What is the total volume (in cubic meters) of the tank?

(b) Suppose the tank is initially full at time $t = 0$, but that after 1 minute there is only 60% of the water remaining. What is the value of $H(2)$?[1]

(c) Using your answer to (b), determine the value of the constant k.

(d) Using the value of k from (c), how long will it take before the tank is empty?

For (a), we calculate the volume by

$$V = \int_1^5 \pi f(x)^2 dx = \pi \int_1^5 \frac{1}{9} x dx = \frac{\pi}{9}(\frac{1}{2}x^2)\Big|_1^5 = \frac{\pi}{9}(\frac{25-1}{2}) = \frac{12\pi}{9}.$$

Thus the volume is $\frac{4\pi}{3}$ cubic meters which is ≈ 4.1888 cubic meters.

For (b), first observe that the area $A_{H(t)}$ of the circular cross section at water-level $H(t)$ is equal to $\pi(f(H(t)+1))^2$. To see this note that the radius of the circle corresponding to the water-level at time t is $f(H(t)+1)$. So, in particular, $A_{H(t)} = \frac{\pi}{9}(H(t)+1)$. Thus,

$$\frac{dH}{dt} = -\frac{ka\sqrt{H(t)}}{A_{H(t)}} = -\frac{ka\sqrt{H(t)}}{\frac{\pi}{9}(H(t)+1)} = -\frac{9k\sqrt{H(t)}}{5\pi(H(t)+1)}.$$

We would like to find $H(1)$ using the fact that the amount of water at time $t = 1$ is 60% of $\frac{4\pi}{3}$ cubic meters. I.e., we would like to determine the y such that

$$\int_1^y \pi f(x)^2 dx = \frac{4\pi}{5}.$$

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[1] This is a typo, the question should ask you to find $H(1)$. To find $H(2)$ one also would need to first find $H(1)$ and k.

Once we have found y, then $H(1) = y-1$. Well, $\int_1^y \pi f(x)^2 dx = \frac{\pi}{18}(y^2-1)$ and so $y = \sqrt{\frac{77}{5}} \approx 3.9243$. Thus, $H(1) = \sqrt{\frac{77}{5}} - 1 \approx 2.9243$ meters.

For (c), we use the fact that, since we are dealing with an autonomous differential equation and H is not an equilibrium, $1 = \int_{H(0)}^{H(1)} \left(-\frac{5\pi(h+1)}{9k\sqrt{h}}\right) dh$. Thus,

$$1 = -\frac{5\pi}{9k}\left(\int_{H(0)}^{H(1)} \sqrt{h} + \frac{1}{\sqrt{h}} dh\right) = -\frac{5\pi}{9k}\left(\frac{2}{3}h^{3/2} + 2\sqrt{h}\right)\Big|_4^{H(1)} \approx \frac{4.5020}{k}$$

and $k \approx 4.5020$.

For (d), we will solve for the time t at which $H(t) = 0$ using the fact that, again since we are dealing with an autonomous differential equation, $t = \int_4^0 \left(-\frac{5\pi(h+1)}{9k\sqrt{h}}\right) dh$. I.e.,

$$t = -\frac{5\pi}{9k}\left(\frac{2}{3}h^{3/2} + 2\sqrt{h}\right)\Big|_4^0 = -\frac{5\pi}{9k}(-\frac{2}{3}4^{3/2} - 4) \approx 3.6183.$$

Thus, after approximately 3.6183 minutes the tank will be empty.