The Fundamental Theorem of Calculus (FTC)

The FTC connects the two notions: definite and indefinite integrals.

$\int f(t)\,dt = F(t) + C$ = a family of all antiderivatives of $f(t)$

$\int_a^b f(t)\,dt = \lim_{n\to\infty} I_R = \lim_{n\to\infty} I_L =$

the limit of Riemann sums

FTC

$F(t) =$ quantity measured at time t

$t =$ time

$a \leq t \leq b$

$f(t) = F'(t) =$ the derivative of $F(t)$

FTC

If $F'(t) = f(t)$ is continuous on $[a, b]$ then the following is true:

$$\int_a^b F'(t)\,dt = \int_a^b f(t)\,dt = F(b) - F(a)$$

The definite integral of the rate of change of some quantity ($\int_a^b F'(t)\,dt$) gives the <u>total</u> <u>change</u> in the quantity between $t=a$ and $t=b$.

$(F(b) - F(a))$

$$\int_a^b F'(t)\,dt = \int_a^b f(t)\,dt \overset{FTC}{=}$$

$$\overset{FTC}{=} \int f(t)\,dt \Big|_{t=a}^{t=b} =$$

$$= \left[F(t) + C\right]\Big|_{t=a}^{t=b} =$$

$$= \left[F(b) + C\right] - \left[F(a) + C\right] =$$

$$= F(b) + C - F(a) - C =$$

$$= F(b) - F(a)$$

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

$F(t)$ = height of a tree at time t

$F'(t)$ = the rate of change of its height

$F(b)-F(a)$ = total change of the tree height between times $t=a$ and $t=b$

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

→ $F(t)$ = the number of infected individuals at time t.

→ $F'(t)$ = the rate of change of number of infected individuals

$F(b) - F(a)$ = total change in number of infected individuals between $t=a$ and $t=b$

Remark $\int_a^b F'(t)\,dt = F(b) - F(a)$

$$F(b) = \underbrace{F(a)}_{\text{the initial value of some quantity}} + \underbrace{\int_a^b F'(t)\,dt}_{\text{the total change of some quantity}}$$

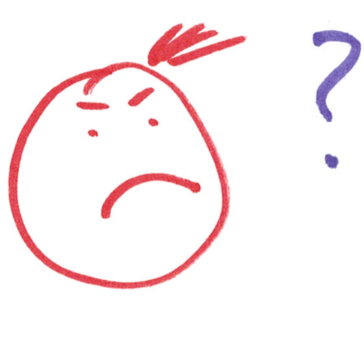
Knowing the LHS, we are able to find the value of the quantity at $t=b$ ($F(b)$)

Example

Let $F(t)$ be the number of bacteria (in millions) at time t.

$F(0)=5$ (Initially, we had $5 \cdot 10^6$ individuals)

The rate of change of the number of bacteria is $3t^2+5t$ (i.e $\frac{dF}{dt} = 3t^2+5t$)



Find

(a) the total increase in the population during the 1-st hour ($F(1) - F(0) = ?$)

(b) the population at $t=1$

($F(1) - ?$)

Using the FTC, we have that the total change in number of bacteria between $t=0$ and $t=1$ is given by

(a) $$\int_0^1 \frac{dF}{dt}\,dt = \int_0^1 (3t^2+5t)dt \overset{FTC}{=} \int (3t^2+5t)dt \Big|_{t=0}^{t=1}$$

$$= (F(t)+C)\Big|_{t=0}^{t=1} = \left(t^3 + \frac{5t^2}{2} + C\right)\Big|_{t=0}^{t=1} =$$

$$= \left(1+\frac{5}{2}+C\right) - (0+0+C) = 1+\frac{5}{2} = 3.5 =$$

$= F(1)-F(0) =$ the total increase in the population during the first hour.

(b) $F(1)$ – ? , where $F(1) =$ the size of population at $t=1$

$$F(1) = F(0) + \int_0^1 (3t^2+5t)dt = 5+3.5 = 8.5.$$

I Table of Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C$$

$$\int \frac{dx}{\sin^2 x} = -\cot x + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \frac{dx}{1+x^2} = \arctan(x) + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$$

$$\int_5^7 (e^t + \cos t - 4t)\,dt =$$

$$= \int (e^t + \cos t - 4t)\,dt \Big|_{t=5}^{t=7} =$$

$$= \left[e^t + \sin t - \frac{4t^2}{2} + C\right]\Bigg|_{t=5}^{t=7} =$$

$$= \left[e^7 + \sin 7 - \frac{4\cdot 49}{2} + C\right] - \left[e^5 + \sin 5 - \frac{4\cdot 25}{2} + C\right]$$

$$= e^7 - e^5 + \sin 7 - \sin 5 - 48.$$

II Substitution technique

$$\int f[g(x)]\cdot g'(x)\,dx =$$

$$= \boxed{\begin{aligned} u &= g(x) \\ du &= g'(x)\,dx \end{aligned}}$$

$$= \int f(u)\,du$$ (see I)

The derivative of the inner function appears in the integrand only up to a multiplicative constant

$$\int \cos(ax+b)\,dx \rightsquigarrow \int \cos x\,dx$$

$$\int \sin(ax+b)\,dx \rightsquigarrow \int \sin x\,dx$$

$$\int e^{ax+b}\,dx \rightsquigarrow \int e^{x}\,dx$$

$$\int \frac{dx}{ax+b} \rightsquigarrow \int \frac{dx}{x}$$

$$x \longrightarrow ax+b$$

a, b = constants

$a \neq 0$

$\int \cos(ax+b)\,dx$ is of the type $\int f(g(x))\,g'(x)\,dx$,

where $\cos(ax+b) = f(g(x))$ is a composite function and $g(x) = ax+b$ is the inner f-n.

$g(x) = ax+b$

~~g(x)=~~ let $u(x) = ax+b$ be a new function

$$\frac{du}{dx} = a$$

$$\frac{du}{a} = dx.$$

Rewrite the original integral in terms of u:

$$\int \cos(ax+b)\,dx = \int \cos(u)\cdot\frac{du}{a} = \frac{1}{a}\int \cos(u)\,du =$$

$$= \frac{1}{a}\sin(u) + C = \frac{1}{a}\sin(ax+b) + C.$$

Similarly, $\int e^{ax+b}\,dx = \dfrac{e^{ax+b}}{a} + C$

$$\int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a} + C.$$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C.$$

$$\int \frac{dx}{x-7} = \boxed{\begin{array}{l} u = x-7 \\ du = dx \end{array}} = \int \frac{du}{u} = \ln|u| + C =$$

$$= \ln|x-7| + C$$

However, $\int \frac{dx}{7-x} = - \ln|7-x| + C$

$$\int \frac{dx}{7-x} = \boxed{\begin{array}{l} u(x) = 7-x \\ du = -dx \end{array}} = \int \frac{-du}{u} =$$

$$= -\int \frac{du}{u} = -\ln|u| + C = -\ln|7-x| + C$$

Check:

$$-\ln|7-x| = \begin{cases} -\ln(7-x), & x<7 \\ -\ln(x-7), & x>7 \end{cases}$$

at $x=7$ $\ln(0) = -\infty$.

$$(-\ln|7-x|)' = \begin{cases} (-\ln(7-x))' = -\left(\frac{1}{7-x}\cdot(-1)\right) = \frac{1}{7-x}, & x<7 \\ (-\ln(x-7))' = -\frac{1}{x-7} = \frac{1}{7-x}, & x>7 \end{cases}$$

$$\int_0^{\frac{\pi}{6}} \cos x \cdot e^{\sin x}\,dx - ?$$

Method 1: $$\int_0^{\frac{\pi}{6}} \cos x\, e^{\sin x}\,dx \overset{FTC}{=} \int \cos x\, e^{\sin x}\,dx \Big|_{x=0}^{x=\frac{\pi}{6}}$$

$$\int \cos x\, e^{\sin x}\,dx = \boxed{\begin{array}{l} u(x) = \sin x \\ \frac{du}{dx} = \cos x \Rightarrow du = \cos x\,dx \end{array}} =$$

$$= \int e^u\,du = e^u + C = e^{\sin x} + C.$$

Thus, $$\int \cos x\, e^{\sin x}\,dx \Big|_{x=0}^{x=\frac{\pi}{6}} =$$

$$= \left(e^{\sin x} + C\right)\Big|_{x=0}^{x=\frac{\pi}{6}} = \left(e^{\sin\frac{\pi}{6}} + C\right) - \left(e^{\sin 0} + C\right) =$$

$$= e^{\frac{1}{2}} - e^{0} = \sqrt{e} - 1$$

Method 2:

$$\int_0^{\frac{\pi}{6}} \cos x\, e^{\sin x}\, dx = \left|\begin{array}{l} \sin x = u(x) \\ \cos x\, dx = du \end{array}\right.$$

x	0	$\frac{\pi}{6}$
u	$\sin 0 = 0$	$\sin\frac{\pi}{6} = \frac{1}{2}$

or

x	0	$\frac{\pi}{6}$
u	0	$\frac{1}{2}$

$$= \int_0^{\frac{1}{2}} e^u\, du \overset{FTC}{=} \int e^u\, du \Big|_0^{\frac{1}{2}} =$$

$$= e^u \Big|_{u=0}^{u=\frac{1}{2}} = e^{\frac{1}{2}} - e^0 = \sqrt{e} - 1.$$

III Integration by Parts

$$\int u \underbrace{dv}_{v'dx} = uv - \int v \underbrace{du}_{u'dx}$$

$$\int P(x)\sin(ax)\,dx$$

$$\int P(x)\cos(ax)\,dx$$

$$\int P(x)e^{ax}\,dx \;,$$

$P(x)$ = polynomial

a = constant $\neq 0$

$$P(x) = u \;;\quad \left.\begin{matrix}\sin(ax)\\ \cos(ax)\\ e^{ax}\end{matrix}\right\} = v'$$

$$\int P(x)\,\underbrace{\ln(ax)}\,dx$$

$$\int P(x)\,\underbrace{\arctan(x)}\,dx$$

$$\int P(x)\,\underbrace{\arcsin(ax)}\,dx$$

$$u(x)$$

$$P(x)\,dx = V'\,dx$$

$$P(x) = V'$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$\int_1^e \sqrt{x}\,\ln x\,dx$$

$\#$

$\ln x = u(x) \implies \frac{dx}{x} = du$

$\sqrt{x}\,dx = v'dx.$

$v' = \sqrt{x}$

$v = \int \sqrt{x}\,dx = \int x^{\frac{1}{2}}dx = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} \Rightarrow$

$v = \frac{2}{3}x^{\frac{3}{2}}$

$$\int_1^e \sqrt{x}\,\ln x\,dx = \underbrace{\frac{2}{3}x^{\frac{3}{2}}}_{v}\cdot\underbrace{\ln x}_{u}\Big|_{x=1}^{x=e} - \int_1^e \underbrace{\frac{2}{3}x^{\frac{3}{2}}}_{v}\,\underbrace{\frac{dx}{x}}_{du} =$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x\Big|_1^e - \frac{2}{3}\int_1^e x^{\frac{1}{2}}dx =$$

$$= \frac{2}{3}e^{\frac{3}{2}}\underbrace{\cancel{\ln e}}_{1} - \frac{2}{3}1^{\frac{3}{2}}\underbrace{\cancel{\ln 1}}_{0} - \frac{2}{3}\cdot\frac{2}{3}x^{\frac{3}{2}}\Big|_1^e =$$

$$= \frac{2}{3}e^{\frac{3}{2}} - 0 - \frac{4}{9}\left(e^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) =$$

$$= \frac{2}{3}e^{\frac{3}{2}} - \frac{4}{9}e^{3/2} + \frac{4}{9} = \frac{2}{3}e^{\frac{3}{2}} + \frac{4}{9}$$