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SEG4145: REAL-TIME AND EMBEDDED SOFTWARE DESIGN

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SOLUTION TO ASSIGNMENT #2

Answer to Question 1

a) Usually, the transfer function is expressed in the Laplace domain. In order to obtain the transfer function as a function of time, one would simply apply the inverse Laplace transform on $H(s)$ in order to obtain $h(t)$.

Let us first determine the transfer function in the Laplace domain. Applying the Laplace transform on the given differential equation, we obtain:

$$2s^2\Theta(s) + 8s\Theta(\mathrm{s}) + 6\Theta(\mathrm{s}) = 2\mathrm{sV_a}(\mathrm{s}) + \mathrm{V_a}(\mathrm{s})$$

... where $\Theta(\mathrm{s}) = \mathcal{L}\{\theta(\mathrm{t})\}$ and $V_a(s) = \mathcal{L}\{\mathrm{v_a}(\mathrm{t})\}$.

Θ is our input; V_a is our output. Therefore:

$$H(s) = \frac{\Theta(\mathrm{s})}{\mathrm{V_a}(\mathrm{s})} = \frac{2s+1}{2s^2+8s+6} = \frac{s+0.5}{s^2+4s+3} = \frac{s+0.5}{(s+1)(s+3)}$$

We perform a partial fraction expansion for this expression:

$$\frac{s+0.5}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$\therefore\ \frac{s+0.5}{s+3} = A + \frac{B(s+1)}{s+3}\ \Bigg|_{s=-1}\ \therefore\ \boxed{A = -0.25}$$

$$\therefore\ \frac{s+0.5}{s+1} = \frac{A(s+3)}{s+1} + B\ \Bigg|_{s=-3}\ \therefore\ \boxed{B = 1.25}$$

Replacing A and B into $H(s)$, we get:

$$H(s) = 1.25\frac{1}{s+3} - 0.25\frac{1}{s+1}$$

In order to obtain the transfer function as a function of time, we apply the inverse Laplace transform:

$$\therefore \boxed{h(t) = 1.25e^{-3t} - 0.25e^{-t}}$$

b) We have that $v_a(t) = t$; therefore, $V_a(s) = \frac{1}{s^2}$.

$$Y(s) = X(s)H(s)$$

$$\therefore \Theta(s) = H(s)V_a(s) = \frac{s+0.5}{s^2(s+1)(s+3)}$$

We perform a partial fraction expansion for the above expression:

$$\frac{s+0.5}{s^2(s+1)(s+3)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1} + \frac{D}{s+3}$$

$$\therefore \frac{s+0.5}{(s+1)(s+3)} = A + Bs + \frac{Cs^2}{s+1} + \frac{Ds^2}{s+3} \quad \Bigg|_{s=0} \therefore \boxed{A = \frac{0.5}{3} = \frac{1}{6}}$$

$$\therefore \frac{s+0.5}{s^2(s+3)} = \frac{A(s+1)}{s^2} + \frac{B(s+1)}{s} + C + \frac{D(s+1)}{s+3} \quad \Bigg|_{s=-1} \therefore \boxed{C = -\frac{0.5}{2} = -\frac{1}{4}}$$

$$\therefore \frac{s+0.5}{s^2(s+1)} = \frac{A(s+3)}{s^2} + \frac{B(s+3)}{s} + \frac{C(s+3)}{s+1} + D \quad \Bigg|_{s=-3} \therefore \boxed{D = \frac{2.5}{18} = \frac{5}{36}}$$

In order to obtain B, we replace A, C, and D in $\Theta(s)$:

$$\frac{s+0.5}{s^2(s+1)(s+3)} = \frac{1}{6s^2} + \frac{B}{s} - \frac{1}{4(s+1)} + \frac{5}{36(s+3)}$$

Because this equation has to hold true for any value of s, we can choose an arbitrary s and substitute it into the above equation ($s \neq 0$). For example, for $s = -0.5 = -\frac{1}{2}$:

$$2B = \frac{2}{3} - \frac{1}{2} + \frac{1}{18} = \frac{12}{18} - \frac{9}{18} + \frac{1}{18} = \frac{4}{18} = \frac{2}{9} \therefore \boxed{B = \frac{1}{9}}$$

Therefore, we obtain the following expression for $\Theta(s)$:

$$\Theta(s) = \frac{1}{6s^2} + \frac{1}{9s} - \frac{1}{4(s+1)} + \frac{5}{36(s+3)}$$

Applying the inverse Laplace transform, we get $\theta(t)$:

$$\theta(t) = \frac{1}{6}u_r(t) + \frac{1}{9}u_s(t) - \frac{1}{4}e^{-t} + \frac{5}{36}e^{-3t}$$

... or

$$\boxed{\theta(t) = \frac{1}{6}t - \frac{1}{4}e^{-t} + \frac{5}{36}e^{-3t} + \frac{1}{9} \qquad , t \geq 0}$$

Answer to Question 2

a)

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 2y(t) = 3x(t) \qquad | \mathcal{L}\{\}$$

$$s^2Y(s) + 2sY(s) + 2Y(s) = 3X(s)$$

$$\therefore\ H(s) = \frac{Y(s)}{X(s)} = \frac{3}{s^2 + 2s + 2}$$

The polynomial in the denominator has no real roots, so we cannot attempt a partial fraction expansion; we use a different formula from the inverse Laplace transform tables:

$$\mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}t \qquad (\zeta < 1)$$

In our case, $\omega_n^2 = 2 \ \therefore\ \omega_n = \sqrt{2}$ and $2\zeta\omega_n = 2 \ \therefore\ \zeta = \frac{1}{\sqrt{2}}$ $\quad (< 1)$. Therefore,

$$H(s) = \frac{3}{2}\frac{2}{s^2 + 2\frac{1}{\sqrt{2}}\sqrt{2}s + \left(\sqrt{2}\right)^2}$$

$$\therefore\ \boxed{h(t) = \mathcal{L}^{-1}\{H(s)\} = 3e^{-t}\sin t}$$

b) We have $Y(s) = H(s)X(s)$ and $X(s) = \mathcal{L}\{\delta(s)\} = 1$.

$$\therefore Y(s) = H(s)$$

$$\therefore \boxed{y(t) = h(t) = 3e^{-t}\sin t}$$

c) Yes, it is going to stabilize. As time approaches infinity, the exponential in the expression of $y(t)$ goes towards zero:

$$\lim_{t\to\infty} y(t) = 0$$

Since this limit exists and is finite in value, this means that the system will stabilize.

Answer to Question 3

a) We have $H_{eq}(s) = \frac{H_1(s)}{1-H_1(s)H_2(s)}$, where $H_1(s) = 2$ and $H_2(s) = \frac{8}{s^2}$.

$$\therefore \boxed{H_{eq}(s) = \frac{2}{1-\frac{16}{s^2}} = \frac{2s^2}{s^2-16}}$$

b) We know that $Y(s) = X(s)H_{eq}(s)$ and $X(s) = \mathcal{L}\{u_s(t)\} = \frac{1}{s}$.

$$\therefore Y(s) = \frac{2s}{s^2-16} = \frac{2s}{(s+4)(s-4)}$$

In the inverse Laplace transform tables, we can find the following formula:

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+\alpha)(s+\beta)}\right\} = \frac{1}{\beta-\alpha}\left(\beta e^{-\beta t} - \alpha e^{-\alpha t}\right) \quad (\alpha \neq \beta)$$

In our case, $\alpha = -4$ and $\beta = 4$ (or the other way around – it makes no difference!)

$$\therefore \boxed{y(t) = \frac{1}{4}(4e^{-4t} + 4e^{4t}) = e^{-4t} + e^{4t}}$$

Answer to Question 4

a) If $H_1(s) = \frac{1}{s+2}$; $H_2(s) = \frac{1}{s+3}$; and $H_3(s) = 4$, then

$$H_{eq}(s) = \frac{H_1(s)H_2(s)}{1 + H_1(s)H_2(s)H_3(s)}$$

Substituting each of the three transfer functions into the above expression, we get:

$$H_{eq}(s) = \frac{\frac{1}{(s+2)(s+3)}}{1 + \frac{4}{(s+2)(s+3)}} = \frac{1}{(s+2)(s+3)+4}$$

$$\therefore \boxed{H_{eq}(s) = \frac{1}{s^2 + 5s + 10}}$$

b) If $x(t) = u_s(t)$, we derive the following expression for the output in the Laplace domain:

$$Y(s) = X(s)H_{eq}(s) = \frac{1}{s(s^2 + 5s + 10)}$$

In this case, since the second-degree polynomial in the denominator does not have real roots, we cannot perform a partial fraction expansion. Instead, we have to find an alternate inverse Laplace transform to apply in order to get $y(t)$.

From the inverse Laplace transform tables, we get the following formula:

$$\mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}\right\} = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \theta\right)$$

... where $\theta = \cos^{-1}\zeta$ and $\zeta < 1$.

In our case, $\omega_n^2 = 10$, therefore $\omega_n = \sqrt{10}$; and $2\zeta\omega_n = 5$, thus $\zeta = \frac{5}{2\omega_n} = \frac{\sqrt{10}}{4}$.

$$\therefore \boxed{y(t) = \frac{1}{10}\left[1 - \frac{2\sqrt{6}}{3} e^{-\frac{5}{2}t} \sin\left(\frac{\sqrt{15}}{2}t + \cos^{-1}\frac{\sqrt{10}}{4}\right)\right]}$$

Answer to Question 5

a) The equivalent transfer function will be:

$$\boxed{H_{eq}(s) = \frac{F_1(s)F_2(s)}{1 + F_1(s)F_2(s)F_3(s)}}$$

b) Replacing $F_1(s) = s$, $F_2(s) = \frac{1}{s+6}$ and $F_3(s) = s + 4$ into $H_{eq}(s)$, we obtain:

$$H_{eq}(s) = \frac{\frac{s}{s+6}}{1 + \frac{s(s+4)}{s+6}} = \frac{s}{s^2 + 5s + 6} = \frac{s}{(s+2)(s+3)}$$

Since $x(t) = \delta(t)$, we have $X(t) = 1$, so the output will be

$$Y(s) = H_{eq}(s) = \frac{s}{(s+2)(s+3)}$$

From the inverse Laplace transform tables, we get that:

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+\alpha)(s+\beta)}\right\} = \frac{1}{\beta - \alpha}\left(\beta e^{-\beta t} - \alpha e^{-\alpha t}\right) \quad (\alpha \neq \beta)$$

If $\alpha = 2$ and $\beta = 3$, we get the expression for $y(t)$:

$$\boxed{y(t) = 3e^{-3t} - 2e^{-2t}}$$

c) Yes, the system will stabilize. As time approaches infinity, $y(t)$ approaches zero:

$$\lim_{t \to \infty} y(t) = 0$$

Since this limit exists and is finite, the system will stabilize.

Answer to Question 6

a) In order to determine the transfer function of this somewhat complex block diagram, let us denote $H_{eq}^{1}(s)$ to be the equivalent transfer function of the part of the diagram comprised of $H_1(s)$, $H_2(s)$, and the positive feedback loop. We will have:

$$H_{eq}^{1}(s) = \frac{H_1(s)}{1 - H_1(s)H_2(s)}$$

Of course, upon contracting this part of the diagram, the equivalent transfer function of the whole system can be written as:

$$H_{eq}(s) = \frac{H_{eq}^{1}(s)H_3(s)}{1 + H_{eq}^{1}(s)H_3(s)H_4(s)}$$

Replacing $H_{eq}^{1}(s)$ into $H_{eq}(s)$, we get:

$$H_{eq}(s) = \frac{\dfrac{H_1(s)H_3(s)}{1 - H_1(s)H_2(s)}}{1 + \dfrac{H_1(s)H_3(s)H_4(s)}{1 - H_1(s)H_2(s)}}$$

$$\therefore \boxed{H_{eq}(s) = \frac{H_1(s)H_3(s)}{1 - H_1(s)H_2(s) + H_1(s)H_3(s)H_4(s)}}$$

b) We can now replace the given expressions for $H_1(s) \dots H_4(s)$ into $H_{eq}(s)$:

$$H_{eq}(s) = \frac{2s}{1 - s(s+1) - 2s\frac{1}{s}}$$

$$\therefore H_{eq}(s) = -\frac{2s}{s^2 + s + 1}$$

Applying the inverse Laplace transform, we then get:

$$h_{eq}(t) = \mathcal{L}^{-1}\{H_{eq}(s)\} = \mathcal{L}^{-1}\left\{-2\frac{s}{s^2 + 2\frac{1}{2}s + 1}\right\}$$

$$\therefore \boxed{h_{eq}(t) = \frac{4\sqrt{3}}{3}e^{-\frac{1}{2}t}\left(\sin\frac{\sqrt{3}}{2}t + \cos^{-1}\frac{1}{2}\right)}$$

c) Since the input is the unit step function, we have $x(t) = u_s(t)$ and $X(s) = \frac{1}{s}$. Therefore, we can write that:

$$Y(s) = X(s)H_{eq}(s) = -\frac{2}{s^2 + s + 1}$$

Applying the inverse Laplace transform, we get:

$$y(t) = \mathcal{L}^{-1}\left\{-2\frac{1}{s^2 + 2\frac{1}{2}s + 1}\right\}$$

$$\therefore \boxed{y(t) = -\frac{4\sqrt{3}}{3}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t}$$