FINAL EXAM-MATH 1300

FALL TERM, 2010

R. Blute & A. Novruzi

Name(Print LEGIBLY)____

I.D. Number____

Instructions- This final examination consists of 10 multiple choice questions worth 3 points each. Your answers to the multiple choice questions must be clearly marked in the squares below. There are also 5 long answer questions worth a total of 70 points. For the long answer questions, you must show your work **on the exam itself** and clearly display your answers.
Do not unstaple these pages.

NO CALCULATORS. NO BOOKS. NO NOTES.

TURN OFF YOUR CELL PHONES AND PUT THEM AWAY.

Multiple Choice Answers:

D	A	C	D
#1	#2	#3	#4
C	E	C	D
#5	#6	#7	#8
	B	A	
	#9	#10	

Question 1- Calculate:

$$\lim_{x\to 2} \frac{x^4 - 16}{x - 2}$$

A) 9 **B)** 8 **C)** 17 **D)** 32 **E)** 72

Solution.

$$\frac{x^4 - 16}{x - 2} = \frac{(x^2)^2 - 4^2}{x - 2} = \frac{(x^2 - 4)(x^2 + 4)}{x - 2} = \frac{(x - 2)(x + 2)(x^2 + 4)}{x - 2} = (x + 2)(x^2 + 4).$$

Hence

$$\lim_{x\to 2} \frac{x^4 - 16}{x - 2} = 32.$$

Question 2- The function $y = f(x)$ is defined implicitly by

$$4xy^2 + 3x^2y = 7$$

Find $\frac{dy}{dx}$ at the point $(1, 1)$.

A) $-\frac{10}{11}$ **B)** $-\frac{13}{7}$ **C)** $\frac{1}{13}$ **D)** $-\frac{2}{17}$ **E)** $\frac{10}{3}$

Solution.

We have

$$\begin{aligned} 4(y^2 + 2xyy') + 3(2xy + x^2y') &= 0, \\ 4(1 + 2y') + 3(2 + y') &= 0, \\ 11y'' &= -10, \\ f'(1) &= -\frac{10}{11}. \end{aligned}$$

Question 3- Find the equation of the tangent line to the graph of $y = \sqrt{8x+1}$ when $x = 3$.

A) $y = \frac{4}{5}x + \frac{3}{5}$ **B)** $y = \frac{4}{5}x + \frac{6}{5}$ **C)** $y = \frac{4}{5}x + \frac{13}{5}$ **D)** $y = \frac{4}{5}x - \frac{3}{5}$ **E)** $y = \frac{4}{5}x + \frac{8}{5}$

Solution.

We have

$$\begin{aligned} x_0 &= 3, \quad y_0 = \sqrt{8 \cdot 3 + 1} = 5, \\ y'(3) &= \frac{8}{2\sqrt{8x+1}}\Big|_{x=3} = \frac{8}{2 \cdot 5} = \frac{4}{5}. \end{aligned}$$

Then, tang. line eq. is

$$\begin{aligned} y &= \frac{4}{5}(x-3) + 5, \\ y &= \frac{4}{5}x + \frac{13}{5}. \end{aligned}$$

Question 4- What are the critical points of the function $f(x) = xe^{2x}$?

A) f has no critical points **B)** $x = 0$ **C)** $x = 0, x = -\frac{1}{2}$ **D)** $x = -\frac{1}{2}$ **E)** $x = \frac{1}{4}, x = -\frac{1}{2}$

Solution.

The function has derivative everywhere. So, the only critical points solve $f'(x) = 0$. But

$$\begin{aligned} f'(x) &= e^{2x} + 2xe^{2x} = e^{2x}(1+2x), \\ f'(x) = 0 &\iff (1+2x) = 0, \\ x &= -\frac{1}{2}. \end{aligned}$$

So, the only C.P. is $x = -1/2$.

Question 5- Calculate:

$$\int_0^{\ln(3)} xe^{2x}\text{dx}$$

A) 0 **B)** 1 **C)** $\frac{9}{2}\ln(3)-2$ **D)** $\frac{1}{2}\ln(3)-3$ **E)** $\frac{5}{2}\ln(3)+1$

Solution.

$$\begin{aligned}
I &= x\frac{1}{2}e^{2x}\Big|_0^{\ln(3)} - \int_0^{\ln(3)} \frac{1}{2}e^{2x}dx \\
\Big(u=x, \quad dv=e^{2x}dx\Big) &= \frac{1}{2}\ln(3)e^{2\ln(3)} - 0 - \frac{1}{2}\cdot\frac{1}{2}e^{2x}\Big|_0^{\ln(3)} \\
\Big(du=dx, \quad v=\frac{1}{2}e^{2x}\Big) &= \frac{1}{2}\ln(3)\cdot 3^2 - \frac{1}{4}(e^{2\ln(3)}-1) \\
&= \frac{9}{2}\ln(3) - \frac{1}{4}(3^2-1) \\
&= \frac{9}{2}\ln(3) - 2.
\end{aligned}$$

Question 6- Evaluate:

$$\int_0^1 \frac{3}{x^2}\text{dx}$$

A) 1 **B)** -1 **C)** 3 **D)** -3 **E)** divergent

Solution.

We have an improper integral as $\frac{1}{x^2}$ has a V.A. at $x=0$. Then

$$\begin{aligned}
I &= \lim_{a\to 0^+}\int_a^1 \frac{3}{x^2}dx \\
&= \lim_{a\to 0^*} 3[-x^{-1}]_a^1 \\
&= \lim_{a\to 0^+} 3(-1+a^{-1}) \\
&= +\infty.
\end{aligned}$$

Hence, the integral is divergent.

Question 7- Suppose $f'(x) = \frac{4}{\sqrt{x}}$ and $f(1) = 3$. Find $f(9)$.

A) 13 **B)** 12 **C)** 19 **D)** 37 **E)** 51

Solution.

$$\begin{aligned} f(x) &= \int f'(x)dx = \int \frac{4}{\sqrt{x}}dx = 8x^{1/2} + C, \\ f(1) &= 8 + C = 3, \\ C &= -5, \\ f(x) &= 8x^{1/2} - 5, \\ f(9) &= 8 \cdot 3 - 5 = 19. \end{aligned}$$

Question 8- If the demand function is $D(x) = -x^2 - x + 25$ and the supply function is $S(x) = x^2 + 2x - 2$, find the producer surplus.

A) 41 **B)** 11 **C)** $\frac{9}{2}$ **D)** 27 **E)** $\frac{4}{3}$

Solution.
Intesection point:

$$\begin{aligned} -x^2 - x + 25 &= x^2 + 2x - 2, \\ 2x^2 + 3x - 27 &= 0, \\ x &= \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot (-27)}}{2 \cdot 2} \\ x_0 &= 3 \quad (\textit{and} \quad -\frac{18}{4}, \ \textit{but not considered as negative}), \\ p_0 &= D(x_0) = -3^2 - 3 + 25 = 13. \end{aligned}$$

Then

$$\begin{aligned} P.S &= \int_0^{x_0} (p_0 - S(x))dx \\ &= \int_0^2 (13 - x^2 - 2x + 2)dx = \left[15x - \frac{1}{3}x^3 - x^2\right]_0^3 \\ &= 45 - 9 - 9 = 27. \end{aligned}$$

Question 9- If $f(x,y) = y^2e^{xy}$, what is $f_{xy}(\ln(2), 1)$?

A) $3\ln(2)$ **B)** $6 + 2\ln(2)$ **C)** $5\ln(2)$ **D)** $5 + \ln(2)$ **E)** $4 + 3\ln(2)$

Solution.

$$\begin{aligned}
f_x(x,y) &= y^2e^{xy}y = y^3e^{xy}, \\
f_{xy}(x,y) &= 3y^2e^{xy} + y^3e^{xy}x = (3y^2 + xy^3)e^{xy}, \\
f_{xy}(ln(2),1) &= (3 + \ln(2))e^{\ln(2)} = (3 + \ln(2))2 \\
&= 6 + 2\ln(2).
\end{aligned}$$

Question 10- If $f(x,y) = e^{3x} + 2x - 2y^3$, how many critical points does $f(x,y)$ have?

A) 0 **B)** 1 **C)** 2 **D)** 3 **E)** 4

Solution.
Solve the system

$$\begin{aligned}
f_x(x,y) = 3e^{3x} + 2 &= 0, \\
f_y(x,y) = -6y^2 &= 0.
\end{aligned}$$

The second equation gives $y = 0$, but the first equation does not have a solution as $e^{3x} + 2 > 2$ for arbitrary x. So, f has 0 C.P.

Long Answer Question 1 (14 points)

Recall that radioactive substances decay exponentially, and that the *half-life* of a radioactive substance is the amount of time it takes for half of the substance to decay. Suppose a radioactive substance has a half-life of 6 years.

- If I begin with 11 grams of the substance, how much will I have after 5 years?
- How long does it take for 9 grams of the substance to decay to 1 gram?

Solution.
Let:

t be the time in years,
A_0 the amount of substance at time $t = 0$, and
$A = A(t)$ the amount of substance at time t.

We know that $A = A_0 b^t$.

$b = ?$ It is given that $A(6) = \frac{1}{2}A_0$. As $A(6) = A_0 b^6$ it follows

$$\frac{1}{2}A_0 = A_0 b^6, \quad b^6 = \frac{1}{2}, \quad b = \left(\frac{1}{2}\right)^{1/6}.$$

Hence

$$A = A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{6}}.$$

a) $A_0 = 11$ then

$$A(5) = 11 \left(\frac{1}{2}\right)^{\frac{5}{6}}.$$

b) Here $A_0 = 9$ and so $A(t) = 9\left(\frac{1}{2}\right)^{\frac{t}{6}}$.

We look for t such that $A(t) = 1$. Hence

$$\begin{aligned} 9\left(\frac{1}{2}\right)^{\frac{t}{6}} &= 1, \\ \left(\frac{1}{2}\right)^{\frac{t}{6}} &= \frac{1}{9}, \\ \frac{t}{6}\ln\left(\frac{1}{2}\right) &= \ln\frac{1}{9}, \\ t &= 6\frac{\ln\frac{1}{9}}{\ln\frac{1}{2}} = 6\frac{\ln(9)}{\ln(2)}. \end{aligned}$$

Long Answer Question 2 (14 points)

Calculate the following two indefinite integrals:

$$I_1 = \int \frac{xe^{x^2}}{\sqrt{e^{x^2}+2}} dx$$

Solution.
we have

$$\begin{aligned} & I_1 &=& \int \frac{\frac{1}{2}du}{u^{1/2}} \\ \left(u = e^{x^2}+2, \quad du = 2xe^{x^2}dx\right) & &=& \frac{1}{2}\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\ & &=& u^{1/2} + C \\ & &=& \sqrt{e^{x^2}+2} + C. \end{aligned}$$

$$I_2 = \int x^2 \ln(x) dx$$

Solution.

$$\begin{aligned} & I_2 &=& \ln(x) \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \frac{1}{x} dx \\ \left(u = \ln(x), \quad dv = x^2 dx\right) & &=& \frac{1}{3}x^3 \ln(x) - \frac{1}{3}\int x^2 dx \\ \left(du = \frac{1}{x}dx, \quad v = \frac{1}{3}x^3\right) & &=& \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C. \end{aligned}$$

Long Answer Question 3 (14 points)

The manager of an apple orchard wants to harvest her apples at a time that will maximize revenue. Currently apples are selling for 90 cents per pound. The price will decrease by 2 cents per pound for each week she waits. Each tree in the orchard now has 33 pounds of apples, but for each week she waits, each tree will gain one additional pound of apples.

- Write a formula for the total revenue for one apple tree.
- How many weeks should she wait to produce the maximum revenue?

Be sure to explain why your answer is an absolute maximum.
Solution.

Let x be the number of weeks counted from the time when one pound sells for 90 cents, and $R(x)$ be the revenue for selling the apples of one tree after x weeks.

After x weeks we have:

the price per pound is: 90 - 2x,
the number of pounds of apples is: 33 + x.

Then we have

$$R(x) = (90 - 2x)(33 + x).$$

Critical points are:

$$\begin{aligned} R'(x) &= -2(33 + x) + (90 - 2x) = -4x + 24 = 0, \\ x &= 6. \end{aligned}$$

As $R''(x) = -4 < 0$ it follows that $x = 6$ is a local maximum of R.

As $x = 6$ is the only C.P. of R in $(0, \infty)$, then the local maximum is in fact global maximum.

Hence, she should sell the apples after $x = 6$ weeks in order to maximize the revenue.

Long Answer Question 4 (16 points)

Consider the two functions:

$f(x) = 4 - x^2$ and $g(x) = 3x$

(a) **(4 points)** Find the intersection points of the graphs of the two functions.

(b) **(6 points)** On the next page, graph these functions, and shade the region between the graphs of f and g for $x \in [0, 4]$.

(c) **(6 points)** Find the area of the shaded region.

Solution.

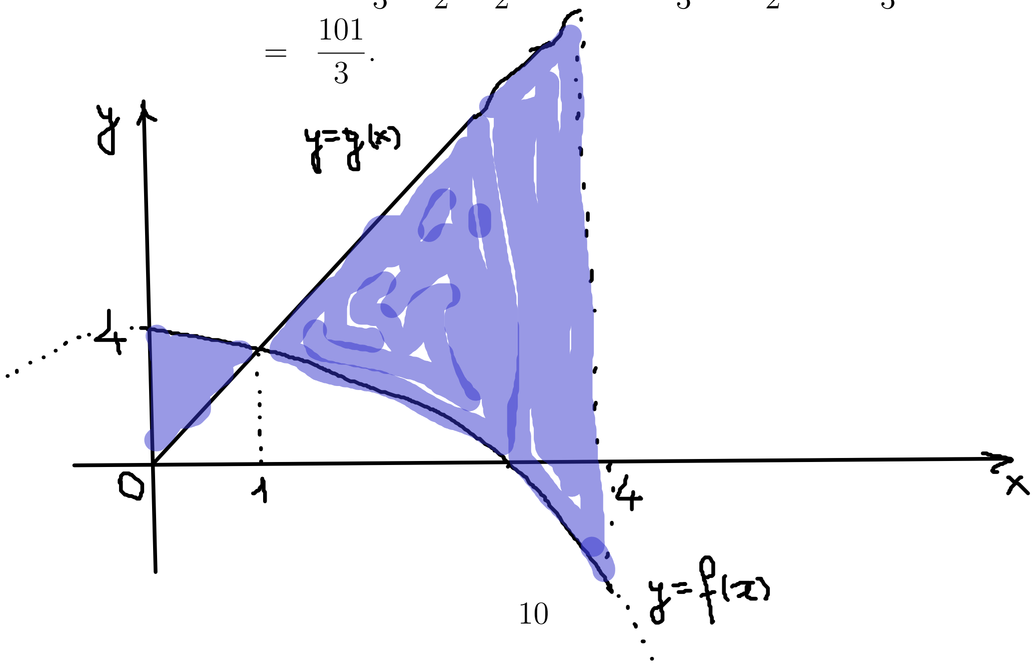
a) Solve

$$\begin{aligned} 4 - x^2 &= 3x, \\ x^2 + 3x - 4 &= 0, \\ x &= \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2} = -4, 1. \end{aligned}$$

b) See the graph.

c) We have

$$\begin{aligned} Area &= \int_0^4 |f(x) - g(x)| dx \\ &= \int_0^1 |f(x) - g(x)| dx + \int_1^4 |(fx) - g(x)| dx \\ &= \int_0^1 (f(x) - g(x)) dx + \int_0^1 (g(x) - f(x)) dx \\ &= \int_0^1 (4 - x^2 - 3x) dx + \int_1^4 (3x - 4 + x^2) dx \\ &= \left[4x - \frac{1}{3}x^3 - \frac{3}{2}x^2\right]_0^1 + \left[\frac{3}{2}x^2 - 4x + \frac{1}{3}x^3 x\right]_1^4 \\ &= 4 - \frac{1}{3} - \frac{3}{2} + \frac{3}{2}4^2 - 4 \cdot 4 + \frac{1}{3}4^3 - \frac{3}{2} + 4 - \frac{1}{3} \\ &= \frac{101}{3}. \end{aligned}$$



Long Answer Question 5 (12 points)

Consider the function of two variables $f(x, y) = \frac{x^3}{3} + y^2 - 3x - 2xy$.

(a) **(3 points)** Calculate the first-order partial derivatives.

(b) **(3 points)** Find all critical points.

(c) **(6 points)** Identify what type of critical points they are (local max, local min or saddle point).

Solution.

a) We have

$$\begin{cases} f_x(x,y) &= x^2 - 3 - 2y, \\ f_y(x,y) &= 2y - 2x \end{cases}$$

b) Solve

$$\begin{cases} f_x(x,y) &= 0, \\ f_y(x,y) &= 0 \end{cases}$$

$$Hence: \quad \begin{cases} x^2 - 2x - 3 &= 0, \\ y &= x \end{cases}$$

$$Therefore \quad \begin{cases} x &= 1 \pm \sqrt{1+3} = -1, 3, \\ y &= -1, 3 \end{cases}$$

So, C.P. are $(-1,-1)$, $(3,3)$.

c) Compute second order derivatives and d:

$$f_{xx}(x,y) = 2x, \quad f_{xy}(x,y) = -2, \quad f_{yy}(x,y) = 2, \qquad d(x,y) = 4x - 4.$$

For $(-1,-1)$:
$d(-1,-1) = -8 < 0$, so $(-1,-1)$ is a S.P.

For $(3,3)$:
$d(3,3) = 8 > 0$, $f_{xx}(3,3) = 6 > 0$, so $(3,3)$ is a point where $f(x,y)$ has a local minimum.

Extra page for additional work