§7.2 #19 Use Euler's method with step sizes

$h = 0.4$, $h = 0.2$, $h = 0.1$

to estimate the value of $y(0.4)$, where y is the solution of the IVP

$$y' = y, \quad y(0) = 1 \qquad (\text{namely } y = e^x).$$

Graph your approximations and find the error in each step size. What happens to the error each time the step size is halved?

19. (a) $y' = F(x, y) = y$ and $y(0) = 1 \quad \Rightarrow \quad x_0 = 0,\ y_0 = 1.$

(i) $h = 0.4$ and $y_1 = y_0 + hF(x_0, y_0) \quad \Rightarrow \quad y_1 = 1 + 0.4 \cdot 1 = 1.4.$ $x_1 = x_0 + h = 0 + 0.4 = 0.4,$

so $y_1 = y\,(0.4) = 1.4.$

(ii) $h = 0.2 \quad \Rightarrow \quad x_1 = 0.2$ and $x_2 = 0.4$, so we need to find y_2.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2y_0 = 1 + 0.2 \cdot 1 = 1.2,$

$y_2 = y_1 + hF(x_1, y_1) = 1.2 + 0.2y_1 = 1.2 + 0.2 \cdot 1.2 = 1.44.$

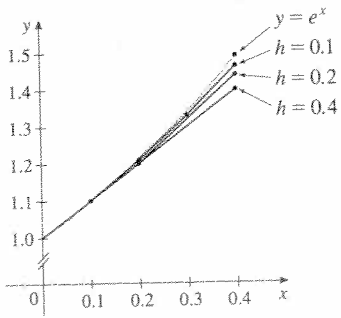
(iii) $h = 0.1 \quad \Rightarrow \quad x_4 = 0.4$, so we need to find y_4. $y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1y_0 = 1 + 0.1 \cdot 1 = 1.1,$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1y_1 = 1.1 + 0.1 \cdot 1.1 = 1.21,$

$y_3 = y_2 + hF(x_2, y_2) = 1.21 + 0.1y_2 = 1.21 + 0.1 \cdot 1.21 = 1.331,$

$y_4 = y_3 + hF(x_3, y_3) = 1.331 + 0.1y_3 = 1.331 + 0.1 \cdot 1.331 = 1.4641.$

(b)



We see that the estimates are underestimates since they are all below the graph of $y = e^x$.

(c) (i) For $h = 0.4$: (exact value) − (approximate value) $= e^{0.4} - 1.4 \approx 0.0918$

(ii) For $h = 0.2$: (exact value) − (approximate value) $= e^{0.4} - 1.44 \approx 0.0518$

(iii) For $h = 0.1$: (exact value) − (approximate value) $= e^{0.4} - 1.4641 \approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

§7.3 #3) $(x^2+1)y' = xy \qquad y(0)=2.$

$$(x^2+1)\frac{dy}{dx} = xy \Rightarrow \frac{1}{y}dy = \frac{x}{x^2+1}dx$$

$$\Rightarrow \ln|y| = \frac{1}{2}\int\frac{2x}{x^2+1}dx = \frac{1}{2}\ln|x^2+1| + C = \frac{1}{2}\ln(x^2+1)+C$$

$$|y| = e^{\ln(\sqrt{x^2+1})}e^{c} = e^{c}\sqrt{x^2+1}$$

$$y = \pm e^{c}\sqrt{x^2+1} \rightsquigarrow y = K\sqrt{x^2+1} \quad (\text{any } K, \text{ incl. } 0)$$

$$2 = y(2) = K\sqrt{2^2+1} = K\sqrt{5} \Rightarrow K = 2/\sqrt{5}$$

$$\boxed{y = \frac{2}{\sqrt{5}}\sqrt{x^2+1}}$$

#17) $y'\tan(x) = a+y, \qquad y(\pi/3) = a, \quad 0 < x < \pi/2$

$$\frac{dy}{dx}\cdot\tan(x) = a+y \Rightarrow \frac{dy}{a+y} = \frac{dx}{\tan x} = \cot x\,dx$$

$$\Rightarrow \ln|a+y| = \ln|\sin x| + C \Rightarrow |a+y| = e^{c}|\sin x| = e^{c}\sin(x)$$

$$a+y = \pm e^{c}\sin(x) = k\sin(x)$$

$$y(\pi/3) = a \Rightarrow 2a = a + y(\pi/3) = k\sin(\pi/3) = k\,\sqrt{3}/2 \Rightarrow k = \frac{4a}{\sqrt{3}}$$

$$a+y = \frac{4a}{\sqrt{3}}\sin(x) \Rightarrow \boxed{y = \frac{4a}{\sqrt{3}}\sin(x) - a.}$$

7.3 #47) A vat with 500 gallons of beer contains 4% alcohol by volume. Beer with 6% alcohol is pumped into the vat at 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after one hour?

<u>Sol'n</u> $y(t)$ = gallons alcohol at time t

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= .06\,\frac{\text{gal}}{\text{gal}} \cdot 5\,\frac{\text{gal}}{\text{min}} - \frac{y}{500}\,\frac{\text{gal}}{\text{gal}} \cdot 5\,\frac{\text{gal}}{\text{min}}$$

$$\frac{dy}{dt} = .3 - .01y\ \frac{\text{gal}}{\text{min}}, \quad y(0) = 20 \ (4\% \text{ of } 500)$$

$$100\,dy = (30-y)\,dt$$

$$\frac{100\,dy}{30-y} = dt \Rightarrow -100\ln|30-y| = t + c$$

$$\Rightarrow \ln|30-y| = -\tfrac{1}{100}t + C$$

$$|30-y| = ke^{-.01t}, \quad y < 30 \ (6\% \text{ of } 500 \text{ is } 30 \text{ gal max})$$

$$30 - y > 0 \Rightarrow |30-y| = 30-y$$

$$\Rightarrow 30 - y = ke^{-.01t}$$

$$y = 30 - ke^{-.01t}; \quad y(0) = 20 \Rightarrow k = 10$$

$$y = 30 - 10e^{-.01t}. \quad y(60) \approx 24{,}512 \text{ gal}$$

$$\text{or} \approx \boxed{4.9\%}$$

§7.4

#15) When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 min. in a 20°C room, its temperature has increased to 10°C.

a) What is the drink's temperature after 50 min.?

b) When will the temp. be 15°C?

$$\frac{dT}{dt} = k(T - T_s) = k(T-20).$$

$y = T - 20$

$y(0) = T(0) - 20 = -15$

so, $\frac{dy}{dt} = ky, \quad y(0) = -15.$

$$\Rightarrow \quad y(t) = y(0)e^{kt} = -15e^{kt}$$

$$T(25) = 10 \quad \Rightarrow y(25) = T(25) - 20 = -10$$

$$-10 = y(25) = -15e^{25k}$$

$$e^{25k} = \frac{2}{3} \Rightarrow 25k = \ln(2/3)$$

$$\Rightarrow k = \frac{1}{25}\ln(2/3) \simeq -0.0162$$

$$T(t) = y(t) + 20 = -15e^{-0.0162t} + 20$$

a) $T(50) = -15\underbrace{e^{-0.0162 \cdot 50}}_{4/9} + 20 = \boxed{13\tfrac{1}{3}\ ^\circ C}$

b) $15 = T(t) = y(t) + 20 = -15e^{\frac{1}{25}\ln(\frac{2}{3})t} + 20$

$$-5 = -15e^{\frac{1}{25}\ln(\frac{2}{3})t} \Rightarrow \frac{1}{3} = e^{\frac{1}{25}\ln(2/3)t}$$

$$\ln(\tfrac{1}{3}) = \tfrac{1}{25}\ln(2/3)\, t \Rightarrow t = \frac{25\ln(1/3)}{\ln(2/3)} \approx \boxed{67.74 \text{ min.}}$$

$\Rightarrow$