MAT 1332, Winter 2013 Assignment 3

Due Friday, February 8, 2:00pm.

Late assignments will **not** be accepted; **nor** will unstapled assignments.

Instructor (circle one): R.Balan O. Vasilyeva M.A. Bahmanian

DGD (circle one): 1 2 3 4

Student Name ____________________ Student Number ____________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ____________________

QUESTION 1. Suppose that a turtle grows at a rate of

$$\frac{dL}{dt} = 5e^{-0.1t},$$

where $L(t)$ is its lenght (in cm) after t years. At birth, the turtle measures 3 cm.

(a) Is this a pure-time differential equation or an autonomous differential equation?

This is a pure-time differential equation.

(b) Find the general solution of this differential equation.

$$L(t) = \int 5e^{-0.1t}dt + C$$
$$= 5\int e^{-0.1t}dt + C$$
$$= 5 \cdot \frac{1}{-0.1} e^{-0.1t} + C$$
$$= -50e^{-0.1t} + C$$

(c) Find the solution that satisfies the given initial condition.

At time $t=0$, $L(0)=3$

$$L(0) = -50\,\underbrace{e^{-(0.1)(0)}}_{1} + C = -50 + C$$

Hence $-50 + C = 3 \Rightarrow \boxed{C = 53}$

The solution is: $L(t) = -50\,e^{-0.1t} + 53$

(d) Determine turtule's lenght after 10 years ($L(10)$) and after 20 years ($L(20)$). How much did the turtule grows between ages $t = 10$ and $t = 20$?

$$L(10) = -50\,e^{-(0.1)(10)} + 53 = -50e^{-1} + 53 \approx 34.61 \text{ cm}$$

$$L(20) = -50\,e^{-(0.1)(20)} + 53 = -50e^{-2} + 53 \approx 46.23 \text{ cm}$$

Between the ages of $t=10$ and $t=20$, the turtle grew

$$L(20) - L(10) = \left(-50e^{-2} + 53\right) - \left(-50e^{-1} + 53\right)$$

$$= -50\left(e^{-2} - e^{-1}\right) \approx 11.62 \text{ cm}$$

(e) Using the fundamental theorem of calculus determine the turtle's growth between ages $t = 10$ and $t = 20$.

$$L(20) - L(10) \underset{\text{F.T.C.}}{=} \int_{10}^{20} L'(t)\,dt = \int_{10}^{20} 5e^{-0.1t}\,dt$$

$$= 5 \int_{10}^{20} e^{-0.1t}\,dt$$

$$= 5 \cdot \frac{1}{-0.1} e^{-0.1t} \Big|_{10}^{20} = -50\left(e^{-(0.1)20} - e^{-(0.1)(10)}\right)$$

$$= -50\left(e^{-2} - e^{-1}\right) \approx 11.62 \text{ cm}$$

QUESTION 2. Consider the following differential equation

$$\frac{dy}{dt} = y^2 + 4$$

(a) Is this differential equation pure or autonomous?

This is an autonomous differential equation

(b) Find the solution that satisfies the initial condition $y(0) = 2$.

We separate the variables:

$\frac{dy}{y^2+4} = dt$. We integrate both sides:

(1) $\boxed{\int \frac{dy}{y^2+4} = \int dt}$. We evaluate $\int \frac{1}{y^2+4}\, dy$:

$\int \frac{1}{y^2+4}\, dy = \int \frac{dy}{4\left(\frac{y^2}{4}+1\right)} = \frac{1}{4} \int \frac{dy}{\left(\frac{y}{2}\right)^2+1}$ use substitution $\boxed{u = \frac{y}{2}}$

$= \frac{1}{4} \int \frac{2du}{u^2+1} = \frac{1}{2} \arctan(u)$ $\quad du = \frac{1}{2} dy$

$= \frac{1}{2} \arctan\left(\frac{y}{2}\right)$. (1) becomes:

$\frac{1}{2} \arctan\left(\frac{y}{2}\right) = t + C$; $\arctan\left(\frac{y}{2}\right) = 2(t+C)$

$\frac{y}{2} = \tan(2(t+C)) \Rightarrow \boxed{y(t) = 2\tan(2(t+C))}$

To find C, we use the initial condition $y(0) = 2$.

$y(0) = 2\tan(2(0+C)) = 2\tan(2C)$

$2 = 2\tan(2C) \Rightarrow \tan(2C) = 1 \Rightarrow 2C = \frac{\pi}{4} \Rightarrow \boxed{C = \frac{\pi}{8}}$

$\Rightarrow$ The solution is: $y(t) = 2\tan\left(2\left(t+\frac{\pi}{8}\right)\right)$

QUESTION 3. A steak is cooked in the oven at a temperature of 200°C, which is kept constant all along the cooking process. The steak's temperature, $T(t)$, obeys Newton's law of cooling, namely, it satisfies the differential equation

$$\frac{dT}{dt} = \alpha(200 - T).$$

(a) If the steak's temperature is 20°C when placed in the oven, solve the differential equation.

$\frac{dT}{200-T} = \alpha\, dt$; $\int \frac{dT}{200-T} = \int \alpha\, dt$; $-\ln|200-T| = \alpha t + C$

$|200-T| = e^{-\alpha t - C}$

$T_0 = 20 < 200 \Rightarrow T < 200 \Rightarrow |200-T| = 200-T$

$200 - T = e^{-\alpha t} e^{-C} = e^{-\alpha t} K$, where $K = e^{-C}$

$\Rightarrow \boxed{T(t) = 200 - Ke^{-\alpha t}}$. To determine K we use $T(0)=20$.

$T(0) = 200 - Ke^{-\alpha \cdot 0} = 200 - K$; $20 = 200 - K \Rightarrow \boxed{K = 180}$

The solution is: $\boxed{T(t) = 200 - 180\, e^{-\alpha t}}$

(b) find α using the fact that after 30 minutes the temperature of the steak is 120°C.

We know that $T(30) = 120$. In our case,

$T(30) = 200 - 180\, e^{-\alpha \cdot 30}$. Hence

$200 - 180 e^{-\alpha \cdot 30} = 120$. We solve for α.

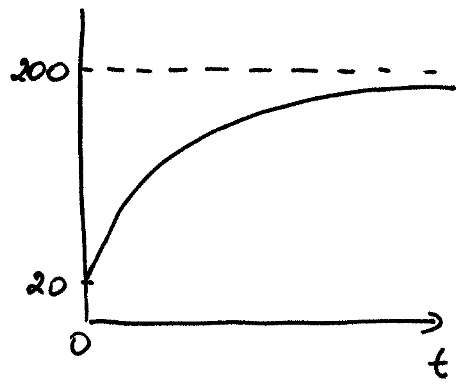
$80 = 180 e^{-\alpha \cdot 30}$

$e^{-\alpha \cdot 30} = \frac{80}{180}\left(= \frac{8}{18} = \frac{4}{9}\right)$. Take logarithm

$-\alpha \cdot 30 = \ln\left(\frac{4}{9}\right)$

$-\alpha = \frac{1}{30} \ln\left(\frac{4}{9}\right) \Rightarrow \boxed{\alpha = -\frac{1}{30} \ln\left(\frac{4}{9}\right)} \approx 0.027$

(c) Using the values found in (a) and (b), plot the graph of the solution $T(t)$.



$T(t) = 200 - 180\,e^{-0.027t}$

QUESTION 4. (a) Find the general solution the separable differential equation

$$y' = e^{-2t}(1+y^2).$$

We separate the variables: $\left(\frac{dy}{dt} = e^{-2t}(1+y^2)\right)$

$\frac{dy}{1+y^2} = e^{-2t}dt$. We integrate both sides.

$\int \frac{dy}{1+y^2} = \int e^{-2t}dt$

$\arctan(y) = -\frac{1}{2}e^{-2t} + C$

$\Rightarrow \boxed{y(t) = \tan\left(-\frac{1}{2}e^{-2t}+c\right)}$

$y(t) = \boxed{\tan\left(-\frac{1}{2}e^{-2t}+c\right)}$, with C an arbitrary constant.

(b) Find the particular solution with initial condition $y(0) = 0$.

$y(0) = \tan\left(-\frac{1}{2}e^{-2\cdot 0}+c\right) = \tan\left(-\frac{1}{2}+c\right)$

$y(0)=0 \Rightarrow \tan\left(-\frac{1}{2}+c\right) = 0 \Rightarrow -\frac{1}{2}+c=0$, i.e.

$\boxed{c=\frac{1}{2}}$

The solution is: $y(t) = \tan\left(-\frac{1}{2}e^{-2t}+\frac{1}{2}\right)$

$= \tan\left(\frac{1}{2}(1-e^{-2t})\right)$

QUESTION 5. (a) Find the general solution of the differential equation

$$y' = y \sin(\pi t).$$

We write the equation as $\frac{dy}{dt} = y \sin(\pi t)$.
We separate the variables: $\frac{dy}{y} = \sin(\pi t)\, dt$.
We integrate both sides:

$$\int \frac{dy}{y} = \int \sin(\pi t)\, dt$$

$$\ln|y| = -\frac{1}{\pi}\cos(\pi t) + C$$

$$|y(t)| = e^{-\frac{1}{\pi}\cos(\pi t) + C} = e^{C} \cdot e^{-\frac{1}{\pi}\cos(\pi t)}$$

$\Rightarrow y(t) = \pm e^{C} \cdot e^{-\frac{1}{\pi}\cos(\pi t)}$. We denote $\pm e^{C} = K$.

The solution is: $\boxed{y(t) = K e^{-\frac{1}{\pi}\cos(\pi t)}}$

$y(t) =$ $K e^{-\frac{1}{\pi}\cos(\pi t)}$, with C an arbitrary constant.

(b) Find the particular solution with initial condition $y(1) = 1$.

$$y(1) = K e^{-\frac{1}{\pi}\cos(\pi)}$$

$$= K e^{-\frac{1}{\pi}\cdot(-1)} = K e^{1/\pi}$$

$y(1) = 1 \Rightarrow K e^{1/\pi} = 1 \Rightarrow \boxed{K = e^{-\frac{1}{\pi}}} \approx 0.73$

The solution is: $y(t) = e^{-\frac{1}{\pi}} \cdot e^{-\frac{1}{\pi}\cos(\pi t)}$

$$= e^{-\frac{1}{\pi}(1 + \cos(\pi t))}$$