MAT 1332: Calculus for Life Sciences

A course based on the book
Modeling the dynamics of life
by F.R. Adler

Supplementary material
University of Ottawa

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April 13, 2010

The inverse tangent function

The tangent function is defined as

$$\tan(x) = \frac{\sin(x)}{\cos(x)},$$

and its derivative can be compute by the quotient rule as

$$\frac{d}{dx}\tan(x) = \frac{\frac{d}{dx}\sin(x)\cos(x) - \sin(x)\frac{d}{dx}\cos(x)}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}.$$

In particular, the function is defined for all x that are not odd multiples of π, and the function is monotone increasing, see Figure 1.

The inverse of the tangent is denoted as arctan or $\tan^{-1}$ and it is defined in the usual way as

$$\arctan(\tan(x)) = x, \qquad \tan(\arctan(x)) = x.$$

See Figure 1. What is its derivative? We differentiate the first equality above by the chain rule and find

$$\frac{d}{dx}[\arctan(\tan(x))] = \frac{d}{dy}\arctan(y)\frac{d}{dx}\tan(x) = 1,$$

with $y = \tan(x)$, since $\frac{d}{dx}x = 1$. Hence, we can divide

$$\frac{d}{dy}\arctan(y) = \frac{1}{\frac{d}{dx}\tan(x)} = \frac{\cos^2(x)}{\cos^2(x)+\sin^2(x)} = \frac{1}{1+\frac{\sin^2(x)}{\cos^2(x)}} = \frac{1}{1+\tan^2(x)} = \frac{1}{1+y^2}.$$

Application to integration

We can now integrate the derivative of the arctan function to get

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C.$$

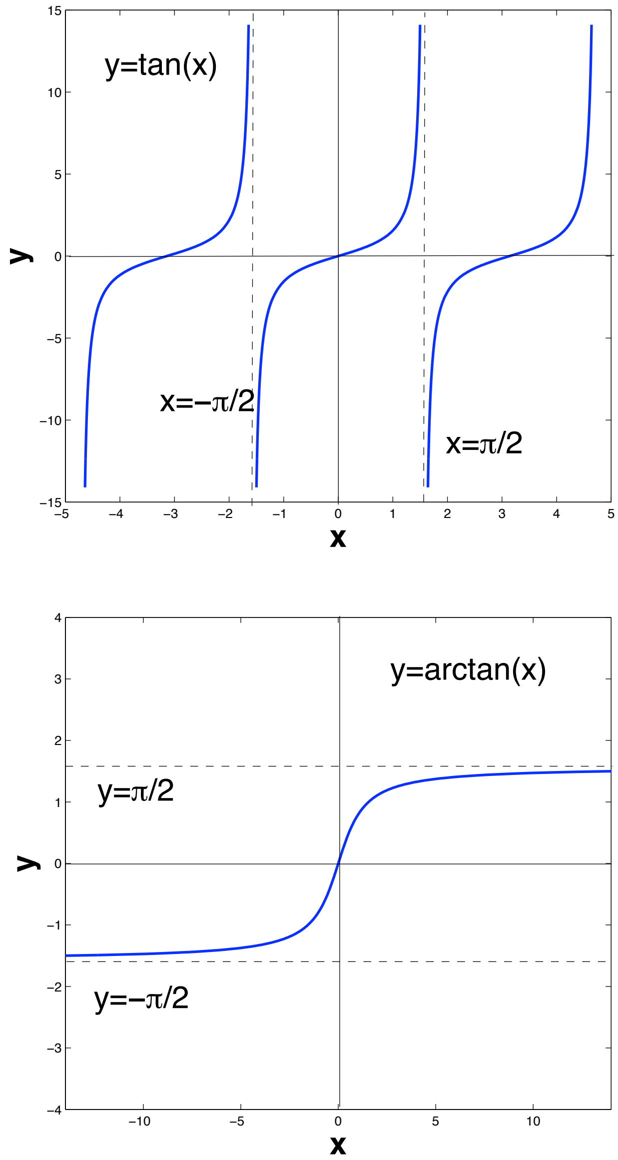

Figure 1: Graphs of the tangent function and its inverse, the arctangent function

Integrals and volumes

First introductory example

The *E. coli* bacterium has a rod-like shape with a cylinder in the middle and two half balls at the ends. What is its volume?
The volume of a cylinder with radius r and height h is $\pi r^2 h$. The volume of a ball with radius r is $4\pi r^3/3$. Now we only have to add the two volumes. For *E. coli*, the data are approximately $r = 0.8 \times 10^{-6}$m and $h = 2 \times 10^{-6}$m. This gives a volume of $V = \pi(1.28 + 0.6825) \times 10^{-18}$m $=$ 6.16×10^{-18}m.

Second introductory example

A termite mound is approximately cone shaped. What is its volume? If we imagine that we cut a cone into thin horizontal slices (perpendicular to its rotational axis), then each slice is approximately a cylinder. For each of these cylinders, we know how to compute the volume; and then we add the volumes. This procedure is very similar to the Riemann sums for the area under a curve, except that we are now using three-dimensional objects. But it gives us the right idea.

General Idea

To calculate the volume of an object with rotational symmetry, we need to know the diameter or radius at each point of its rotational axis. This gives the area of the cut surface at each point. Then we find the volume by integrating the area. See Figure 2.

More precisely, assume that the rotational axis is the x-axis and the radius at each point is given by $f(x)$. Then the area of the cut surface at point x is $A(x) = \pi f^2(x)$ and the volume of the object between points a and b is given by

$$\text{Volume} = V = \int_a^b A(x)dx = \pi \int_a^b (f(x))^2 dx.$$

Example 1: The cylinder

Rotating a constant function $f(x) = r$ around the x-axis gives a cylinder. The volume of the cylinder between $x = 0$ and $x = h$ is

$$V = \int_0^h \pi r^2 dx = \pi r^2 [x]_0^h = \pi r^2 h.$$

Of course, we knew that before.

Example 2: The ball

Rotating the function $f(x) = \sqrt{r^2 - x^2}$ with $-r \leq x \leq r$ around the x- axis gives a ball of radius r. The volume is given by

$$V = \int_{-r}^r \pi(f(x))^2 dx = \int_{-r}^r \pi(r^2 - x^2)dx = \pi \left[r^2 x - \frac{1}{3}x^3\right]_{-r}^r = \frac{4}{3}\pi r^3.$$

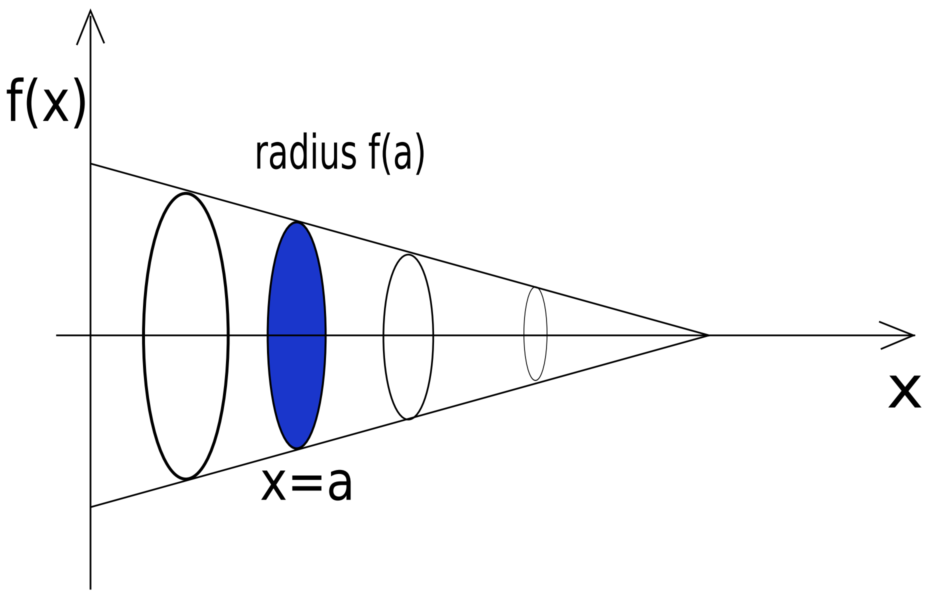

Figure 2: Illustration of volumes of rotation. The straight line function $f(x)$ is rotated around the x-axis. At $x = a$ the radius of the slice is $f(a)$, so the area of the slice is $\pi(f(a))^2$. The volume of the cone is given by integrating the area between the base and the top.

Of course, we know that, too. But now we can even compute the volume of a ball with the top and/or bottom removed. For example, what is the volume of the earth (radius $6.3 * 10^3$ km) with the north and south pole removed (say 500 km on either end)?

$$V = \int_h^h \pi(r^2 - x^2)dx = \pi \left[r^2 x - \frac{1}{3}x^3 \right]_{-h}^h = \pi \left[2r^2 h - \frac{2}{3}h^3 \right].$$

With $r = 6.3 * 10^3$km and $h = 5.8 * 10^3$km we get $V = \pi \left(2(6.3)^2 * 5.8 - \frac{2}{3}(5.8)^3 \right) * 10^9 \text{km}^3 = \pi(460.4040 - 130.0747) * 10^9 \text{km}^3 \approx 10^{10} \text{km}^3$

Example 3: The termite mound

Suppose the cone-shaped termite mound has a base diameter of 1 metre and a height of 2 metres. What is its volume?
At first, we have to find the radius at any given height. At the bottom, the radius is 0.5m, at 2m it is zero. In between, we need a linear function. The function $f(x) = 0.5(1 - x/2)$ gives the profile. Then we integrate to get the volume.

$$V = \pi \int_0^2 (f(x))^2 dx = \frac{\pi}{4} \int_0^2 (1-x/2)^2 dx = \frac{\pi}{4} \int_0^2 (1-x+x^2/4) dx = \frac{\pi}{4} \left[x - x^2/2 + x^3/12 \right]_0^2 = \frac{8\pi}{48} \approx 0.52.$$

The units are, of course, cubic metres.

Example 4: The tree

Suppose that a tree trunk is 25 metres high, and that its radius at height x metre is $r(x) = 2e^{-x}$ metres. What is its volume?

$$V = \pi \int_0^{25} (2e^{-x})^2 dx = 4\pi \int_0^{25} e^{-2x} dx = 2\pi \left[-e^{-2x} \right]_0^{25} = 2\pi(1 - e^{-50}) \approx 2\pi.$$

The units are, again, cubic metres.

Example 5: Trig functions

What is the volume obtained by rotating the function $f(x) = \cos\left(\frac{x}{2}\right)$ with $-\pi \leq x \leq \pi$ around the x-axis? The volume is given by the formula

$$\begin{aligned} V &= \pi \int_{-\pi}^{\pi} [f(x)]^2 \, dx \\ &= \pi \int_{-\pi}^{\pi} \cos^2\left(\frac{x}{2}\right) dx \,. \end{aligned}$$

We can't integrate $\cos^2\theta$ directly, so we have to use our long-forgotten- but-recently-looked-up-in-the-textbook trigonometric identities. Thus

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \qquad (\text{since } \cos^2\theta + \sin^2\theta = 1) \\ &= 2\cos^2\theta - 1 \,. \end{aligned}$$

Thus $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$. Substituting this into the integral, we have

$$V = \pi \int_{-\pi}^{\pi} \frac{\cos 2\left(\frac{x}{2}\right) + 1}{2} dx = \frac{\pi}{2} \int_{-\pi}^{\pi} (\cos x + 1)\, dx = \frac{\pi}{2} \left[\sin x + x \right]_{-\pi}^{\pi}$$

$$= \frac{\pi}{2} \left[(\sin \pi + \pi) - (\sin(-\pi) - \pi) \right] = \frac{\pi}{2}(2\pi) = \pi^2 \text{ units}^3 .$$

Partial fractions

First introductory example

We don't know how to integrate the fraction $\int \frac{x}{x+2}dx$, but we can write the fraction in a simpler way and use known rules to find the integral as follows:

$$\int \frac{x}{x+2}dx = \int \frac{(x+2)-2}{x+2}dx = \int \left[1 - \frac{2}{x+2}\right] dx = x - 2\ln|x+2| + C.$$

Second introductory example

We don't know integrate the fraction $\int \frac{3x-2}{x(x-2)}dx$. But if we simplify the fraction as

$$\frac{3x-2}{x(x-2)} = \frac{1}{x} + \frac{2}{x-2}$$

(check this!) then we can integrate as follows:

$$\int \frac{3x-2}{x(x-2)}dx = \int \left(\frac{1}{x} + \frac{2}{x-2}\right) dx = \ln|x| + 2\ln|x-2| + C.$$

General Idea

Rational functions are fractions of polynomials, i.e., if $P(x)$ and $Q(x)$ are polynomials, then $P(x)/Q(x)$ is called a rational function. We already know how to integrate some of them, namely the following building blocks (you need to know these!)

$$\begin{aligned}
\int \frac{1}{x+a}dx &= \ln|x+a| + C, \\
\int \frac{1}{x^2+1}dx &= \arctan(x) + C = \tan^{-1}(x) + C, \\
\int \frac{x}{x^2+1}dx &= \frac{1}{2}\ln(x^2+1) + C.
\end{aligned}$$

(You don't have to memorize the last one; you could use substitution to solve it.)

If we can split a rational function into sums of these building blocks, then we can integrate easily. The goal of this section is to find a technique to integrate (find antiderivatives of) all rational functions. We only consider cases where $\deg(Q) \leq 2$, i.e., the highest power of x in the denominator is no more than 2. The idea is to decompose a rational function into a sum of simpler rational functions, namely the three examples above, which we know how to integrate.

Recipe for partial fractions

To find the integral of a rational function $P(x)/Q(x)$, follow these steps.

1. If $\deg(P) \geq \deg(Q)$ then use long division to split the rational function into several parts. Now assume that $\deg(P) < \deg(Q)$.

2. If $Q(x) = ax^2 + bx + c = a(x - x_1)(x - x_2)$ has two distinct real roots, the one can find numbers A, B such that
$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A}{x - x_1} + \frac{B}{x - x_2}\right].$$
Then use the natural logarithm to integrate the two terms.

3. If $Q(x) = ax^2 + bx + c = a(x - x_1)^2$ has only one real root, the one can find numbers A, B such that
$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A}{x - x_1} + \frac{B}{(x - x_1)^2}\right].$$
Then one can integrate using substitution, the logarithm, and direct integration.

4. If $Q(x) = ax^2 + bx + c$ has no real roots, then complete the square to get
$$Q(x) = a\left[\left(x - \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] = a[(x - A)^2 + B].$$
Then use the natural logarithm and the arctan to integrate the two terms (potentially substitute first).

We illustrate each of these cases with examples.

Example 1

$P(x) = x^2 + 1, Q(x) = x - 1$. Then $\deg(P) = 2 > 1 = \deg(Q)$, hence we need to do long division. We find
$$x^2 + 1 = (x - 1)(x + 1) + 2.$$
Therefore
$$\int \frac{x^2 + 1}{x - 1} dx = \int \left[x + 1 + \frac{2}{x - 1}\right] dx = \frac{x^2}{2} + x + 2\ln|x - 1| + C.$$

Example 2

$P(x) = 2x^3 + 3x^2 + 2x + 4, Q(x) = x^2 + 1$. Again, since $\deg(P) = 3 > 2 = \deg(Q)$, we need to do long division. We find
$$2x^3 + 3x^2 + 2x + 4 = (x^2 + 1)(2x + 3) + 1.$$
Therefore
$$\int \frac{2x^3 + 3x^2 + 2x + 4}{x^2 + 1} dx = \int \left[2x + 3 + \frac{1}{x^2 + 1}\right] dx = x^2 + 3x + \arctan(x) + C.$$

Example 3

$P(x) = 2x+1, Q(x) = x^2+x-2$. This time, $\deg(P) = 1 < 2 = \deg(Q)$, so no long division is necessary. Instead, we factor Q as $Q(x) = (x-1)(x+2)$, so that

$$\frac{2x+1}{x^2+x-2} = \frac{2x+1}{(x-1)(x+2)}.$$

On the other hand, for two numbers, A, B, we find

$$\frac{A}{x-1} + \frac{B}{x+2} = \frac{(A+B)x + 2A - B}{(x-1)(x+2)}.$$

Comparing with the expression above, we find that $A+B = 2$ and $2A - B = 1$. Hence, $A = B = 1$. Then we integrate

$$\int \left[\frac{2x+1}{x^2+x-2}\right] dx = \int \left[\frac{1}{x-1} + \frac{1}{x+2}\right] dx = \ln|x-1| + \ln|x+2| + C.$$

Example 4

$P(x) = x+5, Q(x) = x^2-4x+4$. Again, $\deg(P) = 1 < 2 = \deg(Q)$, so no long division is necessary. But $Q(x) = (x-2)^2$, has only a single root, i.e.,

$$\frac{x+5}{x^2-4x+4} = \frac{x+5}{(x-2)^2}.$$

On the other hand, for two numbers, A, B, we find

$$\frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{Ax - 2A + B}{(x-2)^2}.$$

Comparing with the expression above, we find that $A = 1$ and $-2A + B = 5$. Hence, $A = 1, B = 7$. Then we integrate

$$\int \left[\frac{x+5}{x^2-4x+4}\right] dx = \int \left[\frac{1}{x-2} + \frac{7}{(x-2)^2}\right] dx = \ln|x-2| - \frac{7}{x-2} + C.$$

Example 5

$P(x) = 3x+2, Q(x) = x^2-2x+5$. No long division necessary. However, Q has no real roots. We complete the square

$$Q(x) = x^2 - 2x + 5 = x^2 - 2x + 1 - 1 + 5 = (x-1)^2 + 4.$$

Now we write

$$\int \frac{3x+2}{x^2-2x+5} dx = \int \frac{3x+2}{(x-1)^2+4} dx = \frac{1}{4} \int \frac{3x+2}{(\frac{x-1}{2})^2+1} dx.$$

This is a case for substitution. We choose $u = \frac{x-1}{2}$ so that $x = 2u + 1$ and $dx = 2du$. Then we get

$$\frac{1}{4}\int \frac{3x+2}{(\frac{x-1}{2})^2+1}dx = \frac{1}{2}\int \frac{6u}{u^2+1}du + \frac{1}{2}\int \frac{5}{u^2+1}du.$$

The first of these integrals requires another substitution, $w = u^2 + 1$, the second is again an acrtan. With this we find

$$\frac{1}{2}\int \frac{6u}{u^2+1}du + \frac{1}{2}\int \frac{5}{u^2+1}du = \frac{1}{2}\int \frac{3}{w}dw + \frac{1}{2}\int \frac{5}{u^2+1}du = \frac{3}{2}\ln|w| + \frac{5}{2}\arctan(u) + C.$$

After back-substituting, we find that the integral with respect to x is given by

$$\frac{3}{2}\ln\left|\frac{x^2}{4} - \frac{x}{2} + \frac{5}{4}\right| + \frac{5}{2}\arctan\left(\frac{x-1}{2}\right) + C.$$

Example 6

$P(x) = x^2 - 2, Q(x) = x^2 - 3x + 2$. Long division first, or the simpler way

$$\frac{x^2-2}{x^2-3x+2} = \frac{x^2-3x+2+3x-4}{x^2-3x+2} = 1 + \frac{3x-4}{x^2-3x+2}.$$

Now, the denominator is $Q(x) = (x-1)(x-2)$, hence we set the partial fractions as

$$\frac{A}{x-1} + \frac{B}{x-2} = \frac{(A+B)x - (A+2B)}{x^2-3x+2}.$$

Hence, we need $A + B = 3$ and $A + 2B = 4$, which is given by $A = 1, B = 2$. Now we can integrate

$$\int \frac{x^2-2}{x^2-3x+2}dx = \int \left(1 + \frac{1}{x-1} + \frac{2}{x-2}\right)dx = x + \ln|x-1| + 2\ln|x-2| + C.$$

Practice Problems

Find the indefinite integral.

1. $\int \frac{1}{(x+3)(x-2)}dx$
2. $\int \frac{1}{x^2-4x+8}dx$
3. $\int \frac{1}{x^2+2x+10}dx$
4. $\int \frac{x-1}{x^2+7x+10}dx$
5. $\int \frac{1}{x^2-16}dx$
6. $\int \frac{1}{x^2+16}dx$

7. $\int \frac{1}{x^2-x-6} dx$

8. $\int \frac{x^2-4x-19}{x^2+5x+6} dx$

9. $\int \frac{x^3+2}{x^2+4} dx$

10. $\int \frac{x^2+9}{x^2-9} dx$

Solutions to Practice Problems

1. $\int \frac{1}{(x+3)(x-2)} dx$
Partial fractions $\frac{1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} = \frac{(A+B)x+3B-2A}{(x+3)(x-2)}$ give $A = -B = -1/5$, and hence we integrate as
$$\int \frac{1}{(x+3)(x-2)} dx = -\frac{1}{5}\left(\frac{1}{x+3} - \frac{1}{x-2}\right) = \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|$$

2. $\int \frac{1}{x^2-4x+8} dx$
Complete the square: $x^2 - 4x + 8 = x^2 - 4x + 4 + 4 = (x-2)^2 + 4$, then integrate
$$\int \frac{dx}{x^2-4x+8} = \int \frac{dx}{(x-2)^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{x-2}{2}\right)^2+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\arctan\left(\frac{x-2}{2}\right)+C.$$

3. $\int \frac{1}{x^2+2x+10} dx$
Complete the square: $x^2 + 2x + 10 = (x+1)^2 + 9$, then integrate
$$\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+9} = \frac{1}{9}\int \frac{dx}{\left(\frac{x+1}{3}\right)^2+1} = \frac{1}{3}\arctan\left(\frac{x+1}{3}\right)+C.$$

4. $\int \frac{x-1}{x^2+7x+10} dx$
The denominator is $x^2 + 7x + 10 = (x+5)(x+2)$. Partial fractions $\frac{A}{x+5} + \frac{B}{x+2} = \frac{(A+B)x+2A+5B}{(x+5)(x+2)}$ give $A = 2, B = -1$ and then we integrate as
$$\int \frac{x-1}{x^2+7x+10} dx = \int \left(\frac{2}{x+5} - \frac{1}{x+2}\right) dx = 2\ln|x+5| - \ln|x+2| + C.$$

5. $\int \frac{1}{x^2-16} dx$
The denominator is $x^2-16 = (x-4)(x+4)$. Partial fractions $\frac{A}{x-4}+\frac{B}{x+4} = \frac{(A+B)x+4(A-B)}{(x-4)(x+4)}$ give $A = -B = 1/8$ so that we can integrate as
$$\int \frac{1}{x^2-16} dx = \frac{1}{8}\int\left(\frac{1}{x-4} - \frac{1}{x+4}\right) dx = \frac{1}{8}\ln\left|\frac{x-4}{x+4}\right|.$$

6. $\int \frac{1}{x^2+16} dx$
This can be integrated directly
$$\int \frac{1}{x^2+16} dx = \frac{1}{16}\int \frac{dx}{\left(\frac{x}{4}\right)^2+1} = \frac{1}{4}\arctan\left(\frac{x}{4}\right) + C.$$

7. $\int \frac{1}{x^2-x-6} dx$
The denominator is $x^2-x-6 = (x-3)(x+2)$. Partial fractions $\frac{A}{x+2}+\frac{B}{x-3} = \frac{(A+B)x-3A+2B}{(x+2)(x-3)}$ give $A = -B = -1/5$ and then we integrate as
$$\int \frac{dx}{x^2-x-6} = -\frac{1}{5}\int\left(\frac{1}{x+2} - \frac{1}{x-3}\right) dx = \frac{1}{5}\ln\left|\frac{x-3}{x+2}\right| + C.$$

8. $\int \frac{x^2-4x-19}{x^2+5x+6} dx$
First of all long division, or write the numerator as $x^2 + 5x + 6 - 9x - 25$ to see that

$$\int \frac{x^2 - 4x - 19}{x^2 + 5x + 6} dx = \int \left(1 - \frac{9x + 25}{x^2 + 5x + 6}\right) dx.$$

Then do partial fractions with $x^2+5x+6 = (x+2)(x+3)$ and $\frac{A}{x+2} + \frac{B}{x+3} = \frac{(A+B)x+3A+2B}{(x+2)(x+3)}$ so that $A = 7$ and $B = 2$. Then integrate as

$$\int \frac{x^2 - 4x - 19}{x^2 + 5x + 6} dx = \int \left(1 - \frac{7}{x+2} - \frac{2}{x+3}\right) dx = x - 7\ln|x+2| - 2\ln|x+3| + C.$$

9. $\int \frac{x^3+2}{x^2+4} dx$
Long division or rewrite the numerator as $x^3 + 2 = x(x^2 + 4) - 4x + 2$. Then (with the substitution $u = x^2 + 4$ in the second term)

$$\int \frac{x^3 + 2}{x^2 + 4} dx = \int \left(x - \frac{4x}{x^2 + 4} + \frac{2}{x^2 + 4}\right) dx = \frac{x^2}{2} - 2\int \frac{du}{u} + \frac{1}{2}\int \frac{dx}{\left(\frac{x}{2}\right)^2 + 1} =$$

$$= \frac{x^2}{2} - 2\ln|x^2 + 4| + \arctan\left(\frac{x}{2}\right) + C.$$

10. $\int \frac{x^2+9}{x^2-9} dx$
First of all the numerator: $x^2+9 = x^2-9+18$ and the denominator $x^2-9 = (x-3)(x+3)$. This gives

$$\int \frac{x^2 + 9}{x^2 - 9} dx = \int \left(1 + \frac{18}{(x-3)(x+3)}\right) dx.$$

Then partial fractions $\frac{A}{x-3} + \frac{B}{x+3} = \frac{(A+B)x+3(A-B)}{(x-3)(x+3)}$. Hence $A = -B = 3$. Then we integrate as

$$\int \frac{x^2 + 9}{x^2 - 9} dx = \int \left(1 + \frac{3}{x-3} - \frac{3}{x+3}\right) dx = x + 3\ln\left|\frac{x-3}{x+3}\right| + C.$$

Complex numbers

Introductory consideration

We can easily solve the equation $x^2 - 4 = 0$. The answer is $x = \pm 2$; in particular, x is a rational number, even an integer. The equation $x^2 - 2 = 0$ is a bit more tricky. The solution $x = \pm\sqrt{2}$ is not a rational number. Instead, we have defined the square root of a positive number as the real number that gives the original number back when multiplied by itself. But what should we do with the equation $x^2 + 1 = 0$? The answer cannot be a real number. (Why?) Can we do the same as above and define a number whose square equals -1? Indeed, this is what mathematicians did in the eighteenth century (it was a daring act and caused a lot of controversy), and they call that number 'i' or *imaginary*. (We will see that complex numbers are hardly more imaginary than $\sqrt{2}$.)

Definition

A *complex number* z is a number of the form

$$z = a + b\mathrm{i}$$

with real numbers a, b and the symbol i that satisfies $\mathrm{i}^2 = -1$. We call $a = \mathrm{Re}(z)$ the *real part* of z and $b = \mathrm{Im}(z)$ the *imaginary part* of z. The real number a can be considered the complex number $a + 0\mathrm{i}$. A complex number of the form $z = b\mathrm{i}$ is called *purely imaginary.*

Addition, subtraction, and multiplication of complex numbers

Complex numbers are easily added, subtracted and multiplied, if we keep the rule $\mathrm{i}^2 = -1$ in mind and use the distributive laws.

$$(a + b\mathrm{i}) \pm (c + d\mathrm{i}) = (a \pm c) + (b \pm d)\mathrm{i}$$

$$(a + b\mathrm{i}) \times (c + d\mathrm{i}) = ac + bc\mathrm{i} + ad\mathrm{i} + bd\mathrm{i}^2 = (ac - bd) + (ad + bc)\mathrm{i}\,,$$

since $\mathrm{i}^2 = -1$.

Examples

1. $(3 + 5\mathrm{i}) + (2 - 7\mathrm{i}) = 5 - 2\mathrm{i}$
2. $(0.5 + 1.7\mathrm{i}) - (0.8 - 2.6\mathrm{i}) = -0.3 + 4.3\mathrm{i}$
3. $(-3 + 2\mathrm{i}) \times (4 - 5\mathrm{i}) = (-12 - (-10)) + (15 + 8)\mathrm{i} = -2 + 23\mathrm{i}$
4. $(2 - 0.5\mathrm{i}) \times (3 + 4\mathrm{i}) = (6 - (-2)) + (-1.5 + 8)\mathrm{i} = 8 + 6.5\mathrm{i}$
5. $(9 + 2\mathrm{i}) + 5 = (9 + 2\mathrm{i}) + (5 + 0\mathrm{i}) = 14 + 2\mathrm{i}$
6. $-3\mathrm{i} + (2 + 3\mathrm{i}) = (0 - 3\mathrm{i}) + (2 + 3\mathrm{i}) = 2 + 0\mathrm{i} = 2$

7. $2 \times (3 - 5\mathrm{i}) = 6 - 10\mathrm{i}$
8. $3\mathrm{i} \times (-1 + 4\mathrm{i}) = -12 - 3\mathrm{i}$

Before we look at inverses and division of complex numbers, we introduce the *complex conjugate* of a complex number.

Definition and observation

The *complex conjugate* of $z = a + b\mathrm{i}$ is $\bar{z} = a - b\mathrm{i}$, i.e., we simply change the sign of the imaginary part. Since the multiplication

$$z\bar{z} = (a + b\mathrm{i})(a - b\mathrm{i}) = a^2 + b^2$$

always produces a non-negative real number, we can take the square root. We define the *modulus* or *absolute value* of $z = a + b\mathrm{i}$ as

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

From the identity $z\bar{z} = |z|^2$, we find the inverse of z to be

$$\frac{1}{z} = z^{-1} = \bar{z}/|z|^2.$$

Example 1

Start with $z = 3 + 4\mathrm{i}$ and $w = 2 - \mathrm{i}$. The complex conjugates are $\bar{z} = 3 - 4\mathrm{i}$ and $\bar{w} = 2 + \mathrm{i}$. The absolute values are $|z| = 5$ and $|w| = \sqrt{5}$. The inverses are

$$z^{-1} = \frac{1}{25}(3 - 4\mathrm{i}), \qquad w^{-1} = \frac{1}{5}(2 + \mathrm{i}).$$

Finally, we can divide

$$\frac{z}{w} = z\frac{\bar{w}}{|w|^2} = \frac{1}{5}(2 + 11\mathrm{i}), \quad \frac{w}{z} = w\frac{\bar{z}}{|z|^2} = \frac{1}{25}(2 - 11\mathrm{i}).$$

Another way to think about this: make the denominator real (similar to the way you'd rationalize the denominator) by multiplying top and bottom by the conjugate of the denominator (i.e., "real-ize" the denominator). Thus,

$$\begin{aligned} \frac{1}{z} &= \frac{1}{3 + 4\mathrm{i}} \\ &= \frac{1}{3 + 4\mathrm{i}} \cdot \frac{3 - 4\mathrm{i}}{3 - 4\mathrm{i}} \\ &= \frac{3 - 4\mathrm{i}}{3^2 - (4\mathrm{i})^2} \\ &= \frac{3 - 4\mathrm{i}}{3^2 + 4^2} \\ &= \frac{3 - 4\mathrm{i}}{25} \end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{w} &= \frac{1}{2+\mathrm{i}} \cdot \frac{2-\mathrm{i}}{2-\mathrm{i}} \\
&= \frac{2-\mathrm{i}}{2^2-\mathrm{i}^2} \\
&= \frac{2-\mathrm{i}}{5} \\
\frac{z}{w} &= \frac{3+4\mathrm{i}}{2-\mathrm{i}} \cdot \frac{2+\mathrm{i}}{2+\mathrm{i}} \\
&= \frac{6+3\mathrm{i}+8\mathrm{i}+4\mathrm{i}^2}{2^2+1^2} \\
&= \frac{2+11\mathrm{i}}{5} \\
\frac{w}{z} &= \frac{2-\mathrm{i}}{3+4\mathrm{i}} \cdot \frac{3-4\mathrm{i}}{3-4\mathrm{i}} \\
&= \frac{6-8\mathrm{i}-3\mathrm{i}+4\mathrm{i}^2}{9+16} \\
&= \frac{2-11\mathrm{i}}{25}
\end{aligned}$$

Example 2

Start with $z = 1-4\mathrm{i}$ and $w = 0.5+3\mathrm{i}$. The complex conjugates are $\bar{z} = 1+4\mathrm{i}$ and $\bar{w} = 0.5-3\mathrm{i}$. The absolute values are $|z| = \sqrt{17}$ and $|w| = \sqrt{37/4}$. The inverses are

$$z^{-1} = \frac{1}{17}(1+4\mathrm{i}), \qquad w^{-1} = \frac{4}{37}(0.5-3\mathrm{i}).$$

Division gives

$$\frac{z}{w} = z\frac{\bar{w}}{|w|^2} = \frac{4}{37}(-11.5-5\mathrm{i}), \quad \frac{w}{z} = w\frac{\bar{z}}{|z|^2} = \frac{1}{17}(-11.5+5\mathrm{i}).$$

Alternatively, we can "real-ize" the denominator as before. Thus

$$\begin{aligned}
\frac{z}{w} &= \frac{1-4\mathrm{i}}{0.5+3\mathrm{i}} \cdot \frac{0.5-3\mathrm{i}}{0.5-3\mathrm{i}} \\
&= \frac{0.5-3\mathrm{i}-2\mathrm{i}+12\mathrm{i}^2}{0.25+9} \\
&= \frac{-11.5-5\mathrm{i}}{9.25} \\
\frac{w}{z} &= \frac{2-\mathrm{i}}{3+4\mathrm{i}} \cdot \frac{3-4\mathrm{i}}{3-4\mathrm{i}} \\
&= \frac{6-8\mathrm{i}-3\mathrm{i}+4\mathrm{i}^2}{9+16} \\
&= \frac{2-11\mathrm{i}}{25}
\end{aligned}$$

Geometric interpretation

It is very helpful to think of a complex number as a point in the plane with the real part as the x-value and the imaginary part as the y-value. Hence, we identify the complex number $z = a + b\mathrm{i}$ with the point (a, b) or with the vector (arrow) from the origin to the point (a, b). (We will talk about vectors in more detail shortly). Then the absolute value of the complex number is simply the distance of the corresponding point from the origin or the length of the vector (arrow). See Figure 3.

With this correspondence, the addition of complex numbers become the addition of vectors as it is known from the physics of forces. See Figure 3.

To interpret multiplication, we take a slightly different point of view. Instead of giving the coordinates of the vector as the endpoint (a, b), we consider its length $r \geq 0$ and the angle ϕ it makes with the x-axis (counterclockwise) as $(r\cos\phi, r\sin\phi)$. This representation is called *polar coordinates*. Then multiplication of two numbers is simply multiplication of the lengths and addition of the angles. See Figure 3. We write

$$z = r(\cos\phi + \mathrm{i}\sin\phi) \quad \text{and} \quad w = s(\cos\psi + \mathrm{i}\sin\psi).$$

Then we multiply, using the trigonometric identities

$$\begin{aligned} zw &= r(\cos\phi + \mathrm{i}\sin\phi) \times s(\cos\psi + \mathrm{i}\sin\psi) \\ &= rs[\cos\phi\cos\psi - \sin\phi\sin\psi + \mathrm{i}(\cos\phi\sin\psi + \cos\psi\sin\phi)] \\ &= rs[\cos(\phi + \psi) + \mathrm{i}\sin(\phi + \psi)]. \end{aligned}$$

Observation and definition

Every complex number of the form $z = \cos\phi + \mathrm{i}\sin\phi$ has absolute value one. We introduce the exponential notation (known as Euler's formula)

$$\exp(\mathrm{i}\phi) = e^{\mathrm{i}\phi} = \cos\phi + \mathrm{i}\sin\phi.$$

It might look strange at first, but the same rules as for the real exponential function apply. In fact, if we denote

$$f(\phi) = \cos\phi + \mathrm{i}\sin\phi$$

then

$$\begin{aligned} f'(\phi) &= -\sin\phi + \mathrm{i}\cos\phi \\ \frac{f'(\phi)}{f(\phi)} &= \frac{-\sin\phi + \mathrm{i}\cos\phi}{\cos\phi + \mathrm{i}\sin\phi} \\ &= \frac{-\sin\phi + \mathrm{i}\cos\phi}{\cos\phi + \mathrm{i}\sin\phi} \cdot \frac{\cos\phi - \mathrm{i}\sin\phi}{\cos\phi - \mathrm{i}\sin\phi} \\ &= \frac{-\sin\phi\cos\phi + \mathrm{i}\sin^2\phi + \mathrm{i}\cos^2\phi + \cos\phi\sin\phi}{\cos^2\phi + \sin^2\phi} \\ &= \mathrm{i} \end{aligned}$$

since $\cos^2\phi + \sin^2\phi = 1$. Now integrate:

$$\begin{aligned} \int \frac{f'(\phi)}{f(\phi)} d\phi &= \int \mathrm{i} d\phi \\ \ln f(\phi) &= \mathrm{i}\phi \\ f(\phi) &= e^{\mathrm{i}\phi} \end{aligned}$$

and thus $\cos\phi + \mathrm{i}\sin\phi = e^{\mathrm{i}\phi}$.

This has many advantages. First of all, we can write any complex number in polar coordinates as $z = re^{\mathrm{i}\phi}$. And we can easily multiply complex numbers in this form. For example, the calculation above becomes a single step (no need to look up the trig identities)

$$re^{\mathrm{i}\phi} \times se^{\mathrm{i}\psi} = rse^{\mathrm{i}(\phi+\psi)}.$$

Examples

1. The complex number $z = 1 + \mathrm{i}$ has modulus $|z| = \sqrt{2}$ and angle $\phi = \pi/4$. Hence $z = 1 + \mathrm{i} = \sqrt{2}e^{\mathrm{i}\pi/4}$.

2. The complex number $w = \sqrt{3} + \mathrm{i}$ has modulus $|w| = 2$ and angle $\phi = \pi/6$. Hence $w = \sqrt{3} + \mathrm{i} = 2e^{\mathrm{i}\pi/6}$.

3. Their product is $zw = (\sqrt{3} - 1) + (\sqrt{3} + 1)\mathrm{i} = 2\sqrt{2}e^{\mathrm{i}5\pi/12}$.

4. In general, if $z = a + b\mathrm{i}$ then $r = |z| = \sqrt{a^2 + b^2}$. The argument ϕ is not uniquely defined. If we restrict it between $-\pi$ and π the we get

$$\begin{cases} \phi = \arctan(b/a) & \text{if} \quad a > 0 \\ \phi = \arctan(b/a) + \pi & \text{if} \quad a < 0, b > 0 \\ \phi = \arctan(b/a) - \pi & \text{if} \quad a < 0, b < 0 \end{cases}$$

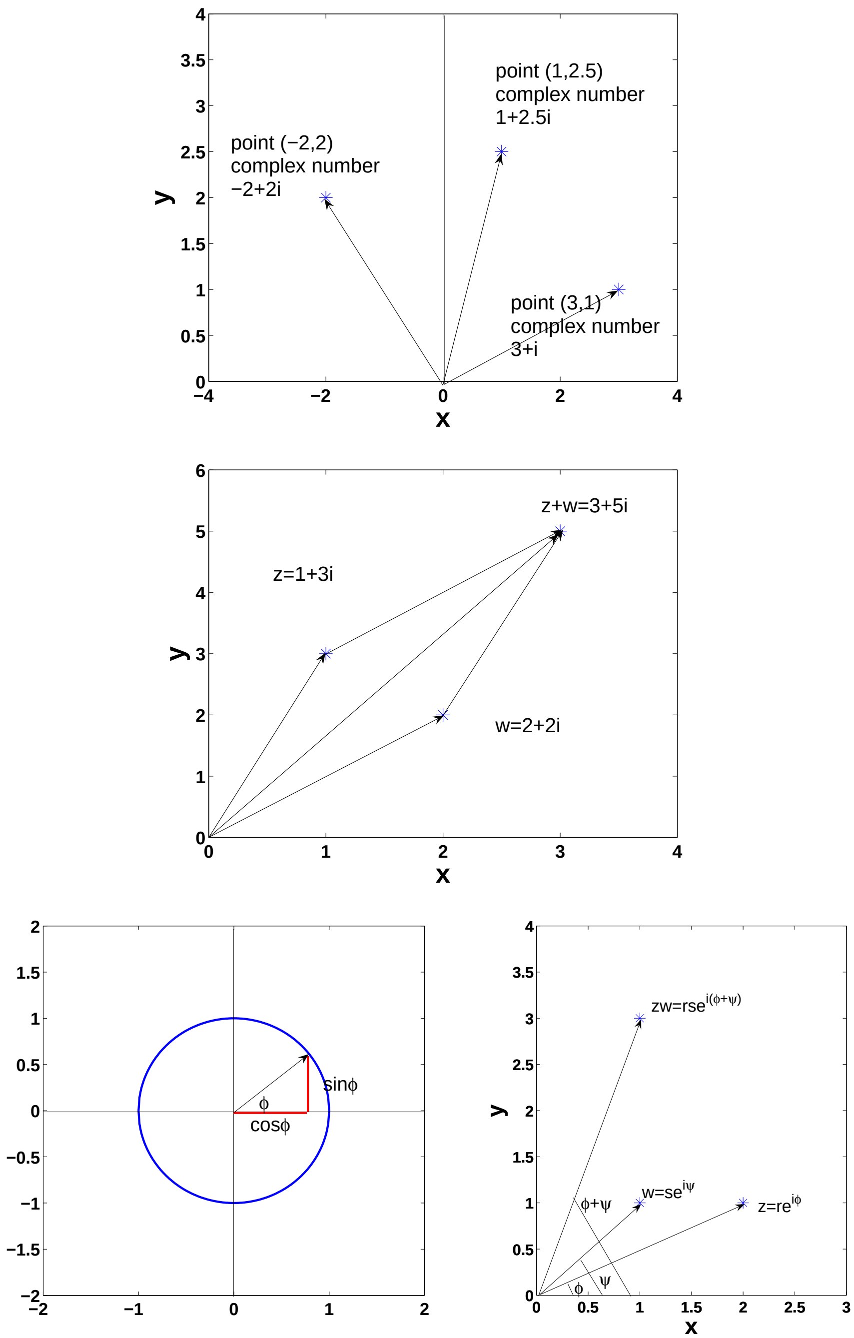

Figure 3: Top panel: correspondence between complex numbers and points in the plane. Middle panel: Addition of complex numbers. Bottom panel: polar coordinates on the unit circle (left) and multiplication of complex numbers using polar coordinates (right)

Linear Algebra I - Linear systems of equations

Introductory example

Suppose there are two types of food. Type I contains 10g of protein and 5g of carbohydrates per 100g, type II contains 8g of protein and 12g of carbohydrates per 100g.

Easy question Suppose you take 50g of type I and 75g of type II, how much protein and carbohydrates do you get? Answer: First we have to choose units, let's say 100g. Then I take 0.5 units of type I and 0.75 units of type II. Now let x_1, x_2 denote the respective units of food type I and II. Then the amount of protein and carbohydrates are given by

$$\begin{aligned} b_1 &= 10x_1 + 8x_2 = 10 \times 0.5 + 8 \times 0.75 = 11, \\ b_2 &= 5x_1 + 12x_2 = 5 \times 0.5 + 12 \times 0.75 = 11.5, \end{aligned}$$

grams respectively. NOTE: we are given x_1 and x_2, and want to find b_1 and b_2.

Harder question Suppose you want to take 16g of protein and 20g of carbohydrates. How much of each food type to you have to take? The equations are just as above.

$$\begin{aligned} 10x_1 + 8x_2 &= b_1 = 16 \\ 5x_1 + 12x_2 &= b_2 = 20. \end{aligned}$$

HOWEVER: this time, with the same notation as above, we are given b_1 and b_2, and want to find x_1 and x_2.
SOLUTION: We multiply the second equation by -2:

$$\begin{aligned} 10x_1 + 8x_2 &= 16 \\ -10x_1 - 24x_2 &= -40. \end{aligned}$$

Then we add the two equations:

$$-16x_2 = -24,$$

which gives $x_2 = 3/2$. This we put back into one of the original equations to get $10x_1 + 12 = 16$ or $x_1 = 2/5$.

The example above is a special case of a *linear system of equations*. More generally, we write a *linear system of m equations with n unknowns* as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned}$$

where

1. $x_1, \ldots, x_n$ are the variables or unknowns,
2. $a_{11}, \ldots, a_{mn}$ are the coefficients,
3. $b_1, \ldots, b_m$ are the right hand side.

A *solution* of such a system is a set of numbers $(s_1, \ldots, s_n)$ that makes *all* the equations true when we substitute the s_i for the x_i. The equations are called *linear* since each of the variables x_i appears only linearly (as opposed to higher powers or other nonlinear functions). The subject of linear algebra is to study such linear systems of equations.

Goal of this chapter

In this chapter we learn how to solve linear systems of equations. In particular, we answer the three questions:

1. Is there always a solution?
2. Can there be more than one solution?
3. How can we compute all solutions?

> **Fact:** A linear system of equations has either
>
> 1. exactly one solution,
> 2. infinitely many solutions, or
> 3. no solution.
>
> Cases (1) and (2) are called *consistent* whereas case (3) is called *inconsistent*.

Examples

Consider the three systems of equations:

$$\left\{ \begin{array}{ll} x_1 - x_2 & = 1 \\ x_1 - x_2 & = 0 \end{array} \right\}, \qquad \left\{ \begin{array}{ll} x_1 - x_2 & = 1 \\ x_1 - 2x_2 & = 0 \end{array} \right\}, \qquad \left\{ \begin{array}{ll} x_1 - x_2 & = 1 \\ -x_1 + x_2 & = -1 \end{array} \right\}.$$

The first system has no solution (since $x_1 - x_2$ cannot simultaneously be both 1 and 0), the second has exactly one solution ($x_1 = 2, x_2 = 1$) and the last system has infinitely many solutions (since the two equations are essentially the same equation): for all real numbers t, the pair $x_1 = t, x_2 = t - 1$ is a solution. We can see graphically why there are these three cases. Each equation is the equation of a line in x_1-x_2 space. In the first case, the two lines are parallel and have no point in common. In the second case, there is one point of intersection. In the third case, the lines are identical, and all points are part of both lines, see Figure 4.

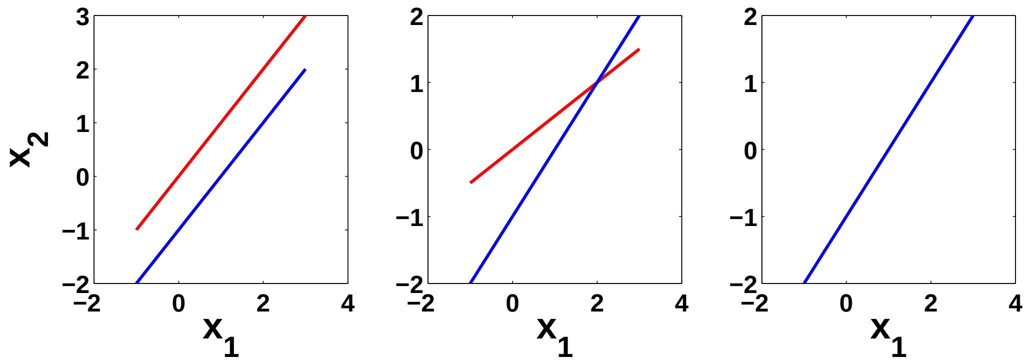

Figure 4: Graphical interpretation of systems of linear equations. Left: no solution, Middle: unique solution, Right: infinitely many solutions.

The Gaussian Elimination Algorithm

Elementary row operations: The solution of a linear system of equations does not change under the following three elementary row operations:

1. multiply a row by a nonzero number,
2. add a multiple of one row to another,
3. exchange the order of two rows.

Example 1

Find the solution of

$$\begin{cases} 2x_1 - x_2 &= 1 \\ 4x_1 + 2x_2 &= 10 \end{cases}$$

Answer:

$$\begin{cases} 2x_1 - x_2 &= 1 \\ 4x_1 + 2x_2 &= 10 \end{cases} \xrightarrow{(-0.5)\times R2} \begin{cases} 2x_1 - x_2 &= 1 \\ -2x_1 - x_2 &= -5 \end{cases} \xrightarrow{R2+R1} \begin{cases} 2x_1 - x_2 &= 1 \\ \quad -2x_2 &= -4 \end{cases}$$

$$\xrightarrow{-2\times R1+R2} \begin{cases} -4x_1 & &= -6 \\ & -2x_2 &= -4 \end{cases} \xrightarrow{(-1/4)\times R1} \xrightarrow{(-1/2)\times R2} \begin{cases} x_1 & &= 3/2 \\ & x_2 &= 2 \end{cases}$$

Now, the solution is obvious: $x_1 = 3/2$ and $x_2 = 2$, or $(3/2, 2)$.

Example 2

Find the solution of

$$\begin{cases} 2x_2 + 3x_3 &= 1 \\ x_1 + x_2 + x_3 &= 2 \\ x_1 - x_2 &= -1 \end{cases}$$

Answer:

$$\left\{\begin{array}{ll} 2x_2+3x_3 &= 1\\ x_1+x_2+x_3 &= 2\\ x_1-x_2 &= -1\end{array}\right. \xrightarrow{R1\leftrightarrow R2} \left\{\begin{array}{ll} x_1+x_2+x_3 &= 2\\ 2x_2+3x_3 &= 1\\ x_1-x_2 &= -1\end{array}\right. \xrightarrow{(-1)\times R3+R1} \left\{\begin{array}{ll} x_1+x_2+x_3 &= 2\\ 2x_2+3x_3 &= 1\\ 2x_2+x_3 &= 3\end{array}\right.$$

$$\xrightarrow{(-1)\times R3+R2} \left\{\begin{array}{ll} x_1+x_2+x_3 &= 2\\ 2x_2+3x_3 &= 1\\ 2x_3 &= -2\end{array}\right. \xrightarrow{(1/2)\times R3} \xrightarrow[R1-R3]{(-3)\times R3+R2} \left\{\begin{array}{ll} x_1+x_2 &= 3\\ 2x_2 &= 4\\ x_3 &= -1\end{array}\right.$$

and finally

$$\xrightarrow[R1-R2]{(1/2)\times R2} \left\{\begin{array}{ll} x_1 &= 1\\ x_2 &= 2\\ x_3 &= -1\end{array}\right.$$

So the solution is $(1,2,-1)$.

Simplify Notation!

Looking at the two examples above, we realize that we do not need to write the variables x_i all the time, provided we agree and stick to a particular order. Similarly, we can do away with the '+' and the '=' signs. The only things that matters are the coefficients in front of the x_i and the right hand side. Hence, we collect these into an array of numbers, which we call *matrix*. We can then simply perform the row operations on the rows of the matrix.

For the example above,

$$\left\{\begin{array}{ll} 2x_2+3x_3 &= 1\\ x_1+x_2+x_3 &= 2\\ x_1-x_2 &= -1\end{array}\right\} \longleftrightarrow \left[\begin{array}{ccc|c} 0 & 2 & 3 & 1\\ 1 & 1 & 1 & 2\\ 1 & -1 & 0 & -1\end{array}\right]$$

We call the matrix above the *augmented matrix*, where we have the coefficients, a_{ij} of the linear system together with the right hand side b_i, separated by the vertical lines. We also call the matrix

$$\left[\begin{array}{ccc} 0 & 2 & 3\\ 1 & 1 & 1\\ 1 & -1 & 0\end{array}\right]$$

the *coefficient matrix*, i.e., this matrix contains only the coefficients a_{ij}.

The same steps as in example 2 above, but in matrix notation are:

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 1\\ 1 & 1 & 1 & 2\\ 1 & -1 & 0 & -1\end{array}\right] \xrightarrow{R1\leftrightarrow R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & 2 & 3 & 1\\ 1 & -1 & 0 & -1\end{array}\right] \xrightarrow{(-1)\times R3+R1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & 2 & 3 & 1\\ 0 & 2 & 1 & 3\end{array}\right]$$

$$\xrightarrow{(-1)\times R3+R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & 2 & 3 & 1\\ 0 & 0 & 2 & -2\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3\\ 0 & 2 & 0 & 4\\ 0 & 0 & 1 & -1\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -1\end{array}\right]$$

Now, we remember that the first column corresponds to x_1, the second to x_2, and the third to x_3. Then the solution is $x_1 = 1, x_2 = 2, x_3 = -1$ or simply $(1,2,-1)$.

Reduced row-echelon form

Now that we have seen several examples of 'simple' forms of systems where the solutions could easily be read off, we will formalize this process a bit.

Definition: The *leading entry* of a row in a matrix is the leftmost nonzero coefficient in that row.
A matrix is in *row-echelon form* if the following three rules are true

1. Rows of zeros are below any nonzero row.
2. The leading entry of any row is to the right of any leading entry in any row above it.
3. All entries in the column below a leading entry are zero.

A matrix is in *reduced row-echelon form* if it is in row-echelon form and *in addition*

4. Each leading entry is 1.
5. All entries in the column above a leading entry are zero.

Examples

The first two systems are in row-echelon form but not reduced. The third and fourth are in reduced row-echelon form, the last one is neither.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix},$$

Example 3

Find the solution of

$$\begin{cases} x_1 + x_2 + x_3 &= 5 \\ 2x_1 \qquad - x_3 &= 0 \\ \qquad x_2 + 4x_3 &= 15 \end{cases}$$

Answer: Use the matrix notation to do the row operations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 0 & -1 & 0 \\ 0 & 1 & 4 & 15 \end{array}\right] \overset{(-2)R1+R2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -2 & -3 & -10 \\ 0 & 1 & 4 & 15 \end{array}\right] \overset{\substack{R2+2R3 \\ (-1)\times R2}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 3 & 10 \\ 0 & 0 & 5 & 20 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Hence, we have the solution $x_1 = 2, x_2 = -1, x_3 = 4$ or $(2, -1, 4)$.

Example 4

Find the solution of

$$\begin{cases} 2x_1 + x_2 + 2x_3 &= 1 \\ -4x_1 \qquad - x_3 &= 2 \\ 2x_1 + 5x_2 + 8x_3 &= 11 \end{cases}$$

Answer in matrix notation:

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ -4 & 0 & -1 & 2 \\ 2 & 5 & 8 & 11 \end{array}\right] \xrightarrow[-R1+R3]{2\times R1+R2} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 4 & 6 & 10 \end{array}\right] \xrightarrow{(-1/2)\times R3+R2} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

At this point, we pause and look at the last row of zeros. In the original notation with variables x_i, this row reads: $0x_1 + 0x_2 + 0x_3 = 2$. This is clearly impossible. Hence, this system is inconsistent, it has no solution.

Example 5

Find the solution of

$$\begin{cases} 2x_1 - 4x_2 - 8x_3 &= -18 \\ 3x_1 + 3x_2 + 15x_3 &= 18 \end{cases}$$

Answer in matrix notation:

$$\left[\begin{array}{ccc|c} 2 & -4 & -8 & -18 \\ 3 & 3 & 15 & 18 \end{array}\right] \xrightarrow{\substack{0.5\times R1 \\ 1/3\times R2}} \left[\begin{array}{ccc|c} 1 & -2 & -4 & -9 \\ 1 & 1 & 5 & 6 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{ccc|c} 1 & -2 & -4 & -9 \\ 0 & 3 & 9 & 15 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & -4 & -9 \\ 0 & 1 & 3 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

This time, the last row reads

$$x_2 + 3x_3 = 5.$$

Whatever we choose for x_3, we can always find an x_2 to make the equation true. Hence, this equation has infinitely many solutions. We set $x_3 = t$ as a *free variable*, then we get $x_2 = 5 - 3t$. We plug this into the first equation and solve for x_1 as

$$x_1 + 2t = 1, \qquad \text{or} \qquad x_1 = 1 - 2t.$$

Hence, the infinitely many solutions can be written as the set $\{(1 - 2t, 5 - 3t, t) \colon t \in \mathbb{R}\}$.

Example 6

Find the solution of

$$\begin{cases} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{cases}$$

Answer in matrix notation:

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right] \overset{\substack{0.5\times R1 \\ R1+R2 \\ -4\times R1+R3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right] \overset{R2+R3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Again, we have a row of zeros at the bottom. However, this time the last row reads: $0x_1 + 0x_2 + 0x_3 = 0$. This equation is satisfied for all values of x_1, x_2, x_3. We do not run into the same problem as in example 4. We simply continue as in example 5.

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3/7 & -1/7 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The second row

$$x_2 + (4/7)x_3 = 1/7.$$

has again infinitely many solutions. We denote $x_3 = t$ as the *free variable* and compute $x_2 = 1/7 - (4/7)t$. In the first row, we get

$$x_1 + (3/7)t = -1/7 \qquad \text{or} \qquad x_1 = -1/7 - 3t/7.$$

Hence, the solution set is $\{(-\frac{1}{7} - \frac{3t}{7}, \frac{1}{7} - \frac{4t}{7}, t) \colon t \in \mathbb{R}\}$.

Example 7

For which values of h does the system

$$\begin{cases} x_1 + hx_2 & = -3 \\ -2x_1 + 4x_4 & = 6 \end{cases}$$

have (a) a unique solution, (b) infinitely many solutions, and (c) no solution?

Answer:

$$\left[\begin{array}{cc|c} 1 & h & 3 \\ -2 & 4 & 6 \end{array}\right] \overset{2\times R1+R2}{\longrightarrow} \left[\begin{array}{cc|c} 1 & h & 3 \\ 0 & 2h-4 & 0 \end{array}\right]$$

The last row gives the equation

$$(2h-4)x_2 = 0.$$

If $2h - 4 = 0$, (i.e., $h = 2$) then $x_2 = t$ is a free variable, and $x_1 = -3 - hx_2 = -3 - 2t$. If, on the other hand, $2h - 4 \neq 0$ then the only way to satisfy the second row is $x_2 = 0$. In this case, the first row gives $x_1 = -3$. Hence, if $h \neq 2$ then there is a unique solution, if $h = 2$ then there are infinitely many solutions. For no value of h is there no solution.

Practice Problems

1. Solve the following systems of equations by bringing them into reduced row-echelon form

$$(a)\quad \begin{cases} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{cases} \qquad (b)\quad \begin{cases} 5x_1 - 2x_2 + 6x_3 &= 0 \\ -2x_1 + x_2 + 3x_3 &= 1 \end{cases}$$

$$(c)\quad \begin{cases} 2x_1 + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{cases} \qquad (d)\begin{cases} 7x_1 + 2x_2 + x_3 - 3x_4 &= 5 \\ x_1 + 2x_2 + 4x_3 &= 1 \end{cases}$$

$$(e)\quad \begin{cases} 3x_1 + 2x_2 - x_3 &= -15 \\ 3x_1 + x_2 + 3x_3 &= 11 \\ -6x_1 - 4x_2 + 2x_3 &= 30 \end{cases} \qquad (f)\quad \begin{cases} 2x_1 - x_2 - 3x_3 &= 0 \\ -x_1 + 2x_2 - 3x_3 &= 0 \\ x_1 + x_2 + 4x_3 &= 0 \end{cases}$$

2. Problems with parameters

(a) For which values of a, b does the system

$$\begin{cases} x_1 + ax_2 &= 1 \\ 2x_1 + 3x_2 &= b \end{cases}$$

have (i) a unique solution, (ii) infinitely many solutions, or (iii) no solutions?

(b) Explain why the system

$$\begin{cases} x_1 + x_2 + 2x_3 &= a \\ x_1 + x_3 &= b \\ 2x_1 + x_2 + 3x_3 &= c \end{cases}$$

is consistent if $c = a + b$ but inconsistent in all other cases.

3. Application

Insects of two species are reared on two types of food. Species 1 consumes 5 units of food A and 3 units of food B per day. Species 2 consumes 2 units of A and 4 units of B, respectively. Every day, 900 units of food A and 960 units of food B are provided. How many individuals of each species are reared?

Solutions to practice problems

1.

$$(a)\quad (1,2,3) \qquad (b)\quad \{(2-12t, 5-27t, t)\colon t\in\mathbb{R}\} \qquad (c)\quad \left(\frac{11}{12}, -\frac{71}{12}, -\frac{5}{12}\right)$$

$$(d)\quad \left\{\left(\frac{4}{3}+\frac{1}{2}s+\frac{1}{2}t, -\frac{1}{6}-\frac{27}{12}s-\frac{1}{4}t, s, t\right) : s,t\in\mathbb{R}\right\} \qquad (e)\quad \left\{\left(\frac{37}{3}-\frac{7}{3}t, 4t-26, t\right) : t\in\mathbb{R}\right\}$$

$$(f)\quad (0,0,0)$$

2.

(a) Unique solution if $a \neq 3/2$. infinitely many solutions if $a = 3/2$ and $b = 2$. No solution if $a = 3/2$ and $b \neq 2$.

(b) Add the first and second row.

3.

Let x_i be the number of individuals of species i. Then the system is

$$5x_1 + 2x_2 = 900, \qquad 3x_1 + 4x_2 = 960,$$

and the solution is $(x_1, x_2) = (120, 150)$.

Linear Algebra II - Vectors and matrices

In the last section, we introduced matrices to simplify our life, as a short hand notation for linear systems of equations. In this section, we study matrices as objects in their own right. We learn when it is possible to add and multiply them and how to do it. While there is no direct biological application in this section, the content presented here is the foundation of everything to come. It is like learning the grammar of a language so that one can speak it properly later.

Definition: An $m \times n$-matrix A is a rectangular array of numbers with m rows and n columns, i.e.,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The numbers a_{ij} are called *entries*.
If $m = n$ then A is a *square matrix*.
A $1 \times n$-matrix is called a row vector: $[c_1, c_2, \dots, c_n]$.
An $m \times 1$-matrix is called a column vector: $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.
Two matrices $A = [a_{ij}], B = [b_{ij}]$ are said to be *equal* if they have the same dimension and if for all i, j we have $a_{ij} = b_{ij}$.

We note two special (classes of) matrices, the *zero matrix*

$$0 = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix},$$

and the *identity matrix*

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

The number of rows and columns of these two matrices is usually clear from the context.
If A is a square matrix, then the elements a_{ii} are called the *diagonal elements* and their sum of called the *trace* of A, i.e.,

$$A = \begin{bmatrix} 5 & 2 & -1 \\ 0 & 1 & 5 \\ 8 & 2 & -4 \end{bmatrix}$$

has the diagonal elements 5, 1, -4 and the trace is $\text{tr}(A) = 5 + 1 - 4 = 2$.

Matrix addition and scalar multiplication

If two matrices, $A = [a_{ij}]$ and $B = [b_{ij}]$ are both of the format $m \times n$, then we can form the sum of the two by entrywise addition to obtain a matrix of the same format

$$C = A + B = [c_{ij}] = [a_{ij} + b_{ij}].$$

If k is a number and $A = [a_{ij}]$ an $m \times n$-matrix, then we define the entrywise product

$$kA = [ka_{ij}],$$

which is again an $m \times n$-matrix.

Examples

Consider the two 2×3-matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then we have

$$A + B = \begin{bmatrix} 2 & 2 & 4 \\ -3 & -1 & -1 \end{bmatrix} \qquad 5A = \begin{bmatrix} 5 & 10 & 15 \\ -15 & -10 & -5 \end{bmatrix},$$

$$3A + 2B = \begin{bmatrix} 3 & 6 & 9 \\ -9 & -6 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 11 \\ -9 & -4 & -3 \end{bmatrix}.$$

The transpose of a matrix

The *transpose* of a matrix A is obtained from A by interchanging rows and columns, or, loosely speaking, by flipping the matrix along its diagonal. More formally, if $A = [a_{ij}]$, then the transpose is

$$A^T = [a_{ji}].$$

If A is of the format $m \times n$ then A^T is of the format $n \times m$. In particular, the transpose of a column vector is a row vector and vice versa.

Examples

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$$

$$v = \begin{bmatrix} -4 \\ 7 \\ 0 \end{bmatrix} \qquad v^T = \begin{bmatrix} -4 & 7 & 0 \end{bmatrix}.$$

Matrix-vector multiplication

If the matrix A has n columns and the column vector x has n rows, then we can define the product Ax as the following column vector:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}.$$

To remember this definition, simply think about linear systems of equations!
Note that the resulting vector has the same number of rows as the matrix. In general, the dimensions work as follows:

$$(m \times n) * (n \times 1) = (m \times 1).$$

Examples

1.
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \times 5 + 2 \times 6 \\ 3 \times 5 + 4 \times 6 \end{bmatrix} = \begin{bmatrix} 5 + 8 \\ 12 + 24 \end{bmatrix} = \begin{bmatrix} 13 \\ 39 \end{bmatrix}$$

2.
$$\begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times (-1) + 3 \times 2 \\ -3 \times 1 + (-2) \times (-1) + (-1) \times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 - 2 + 6 \\ -3 + 2 - 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

3.
$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 + 2 - 3 - 4 \\ 0 + 2 - 0 + 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

4.
$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 3 + 2 \end{bmatrix} = 0$$

5.
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1-2 \\ -3-4 \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \end{bmatrix}$$

6.
$$\begin{bmatrix} -3 & 2 \\ -7 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -15-6 \\ -35-12 \\ 5-18 \end{bmatrix} \begin{bmatrix} -21 \\ -47 \\ -13 \end{bmatrix}$$

7.
$$\begin{bmatrix} 30 & 25 \end{bmatrix} \begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 60+225 \end{bmatrix} = 285$$

Matrix-matrix multiplication

We use the definition of the matrix-vector product to define a product of two matrices. Consider a matrix A with n columns and a matrix B with n rows. We may think of each column of B as a column vector of length n. We know how to multiply each of these with the matrix A and then we put the resulting vectors into one matrix. More formally, the product of

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & \dots & b_{1k} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nk} \end{bmatrix} = \left[\begin{array}{c|c|c|c} B_1 & B_2 & \dots & B_k \end{array} \right]$$

is given by

$$AB = \left[\begin{array}{c|c|c|c} AB_1 & AB_2 & \dots & AB_k \end{array} \right].$$

Note that the dimensions multiply as follows:

$$(m \times n) * (n \times k) = (m \times k).$$

Examples

Take the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 4 & 2 \\ 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}.$$

The product AB is defined and it is

$$\begin{aligned} AB &= \begin{bmatrix} 1\times 1 + 2\times(-1) & 1\times 2 + 2\times 0 & 1\times 1 + 2\times 4 \\ 3\times 1 + 4\times(-1) & 3\times 2 + 4\times 0 & 3\times 1 + 4\times 4 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 2 & 9 \\ -1 & 6 & 19 \end{bmatrix}. \end{aligned}$$

The product BA is not defined since the number of columns of B does not equal the number of rows of A (try it and see!). However, if we transpose B first, then we can multiply $B^T A$:

$$\begin{bmatrix} -2 & -2 \\ 2 & 4 \\ 13 & 18 \end{bmatrix}.$$

The product CA is defined, but not vice versa (check the format!):

$$CA = \begin{bmatrix} -2 & -2 \\ 10 & 16 \\ 9 & 12 \end{bmatrix}.$$

Again, if we transpose C, then we can multiply AC^T

$$AC^T = \begin{bmatrix} -1 & 8 & 6 \\ -1 & 20 & 12 \end{bmatrix}.$$

Since A is a square matrix, we can form the power

$$A^2 = AA = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}.$$

Since A and D are square of the same size, we can multiply them both ways:

$$AD = \begin{bmatrix} -1 & 0 \\ -3 & -2 \end{bmatrix}, \quad DA = \begin{bmatrix} -7 & -10 \\ 3 & 4 \end{bmatrix}.$$

Thus $AD \neq DA$, even though both are defined.

NOTE: The order of the product matters! Matrix multiplication is NOT commutative, even if both products are defined.

Multiplication of the identity matrix and the zero matrix are just as easy as multiplying the real numbers zero and one:

$$AI = IA = A, \qquad A0 = 0A = 0.$$

As a final, not so obvious example for matrix multiplication, we note (check the formats!)

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & -3 & -2 \\ 2 & 6 & 4 \end{bmatrix}.$$

Note: We have seen that many operations with real numbers (addition, multiplication) also work with (some) matrices. Two of several differences are

1. The commutative law does not hold.

2. The cancellation law does not hold.

Example

The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is an example where $A^2 = 0$ but $A \neq 0$.

Practice Problems

Consider the following matrices:

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \qquad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

1. Compute the following if possible. If not, explain why.

(a) $D+E$	(b) $D-E$	(c) $5A$	(d) $-7C$
(e) $2B-C$	(f) $4E-2D$	(g) $-3(D+2E)$	(h) $A-A$
(i) $\text{tr}(D)$	(j) $\text{tr}(D-3E)$	(k) $\text{tr}(A)$	(l) $\text{tr}(B)$
(m) $2A^T+C$	(n) D^T-E^T	(o) $(D-E)^T$	(p) $B-B^T$

2. Compute the following if possible. If not, explain why.

(a) AB	(b) BA	(c) $3ED$	(d) $(AB)C$	(e) $A(BC)$
(f) CC^T	(g) $(DA)^T$	(h) $(C^TB)A^T$	(i) $\text{tr}(DD^T)$	(j) B^3

Solutions to Practice Problems

1.

(a) $D+E = \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$, (b) $D-E = \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$, (c) $5A = \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$,

(d) $-7C = \begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$, (e) $2B-C$ not defined, (f) $4E-2D = \begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$,

(g) $-3(D+2E) = \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$, (h) $A-A = 0$, the zero matrix, (i) $\mathrm{tr}(D) = 1+0+4 = 5$,

(j) $\mathrm{tr}(D-3E) = -17-3-5 = -25$, (k) $\mathrm{tr}(A)$ not defined, (l) $\mathrm{tr}(B) = 4+2 = 6$.

(m) $2AT+C = \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$, (n) $D^T-E^T = \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$, (o) $(D_E)^T = D^T-E^T$,

(p) $B-B^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

2.

(a) $AB = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$, (b) BA not defined, (c) $3ED = \begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$,

(d) $(AB)C = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$, (e) $A(BC) = (AB)C$, (f) $CC^T = \begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix}$,

(g) $(DA)^T \begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$, (h) $(C^T B)A^T = \begin{bmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{bmatrix}$, (i) $\mathrm{tr}(DD^T) = 30+2+29 = 61$,

(j) $B^3 = \begin{bmatrix} 64 & -28 \\ 0 & 8 \end{bmatrix}$.

Linear Algebra III - Inverses and Determinants

We know that, for every nonzero real number, x, there exists an inverse, $x^{-1} = 1/x$ such that the product $xx^{-1} = 1$. In the last section, we learned how to multiply matrices (and when this is possible). Here we ask whether inverses exist for matrices as well, and how we can compute them.

Introductory example

The multiplication of two square matrices can give the identity matrix:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

But of course, not every (square) matrix has an inverse. The zero matrix, for example, is not invertible.

Definition: A square matrix A is *invertible* if there exists another square matrix B of the same dimension, such that

$$AB = I = BA.$$

We then write $A^{-1} = B$.

Note: If a matrix is not square, then it cannot be invertible.

How can we find out whether a given matrix has an inverse and what the inverse is? Let's go back to the example above with

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}.$$

We want to find a matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

such that

$$AB = \left[A \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \middle| A \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, we have to solve the two systems

$$A \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can do this simultaneously, using the three allowed elementary row operations of multiplication, addition and interchange.

$$\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R1-2R2} \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right] \xrightarrow{R1+5R2} \left[\begin{array}{cc|cc} 2 & 0 & 6 & -10 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[-R2]{0.5R1,} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

Once we have reduced the left hand side to the identity matrix, we have the inverse of the original matrix on the right hand side.

Algorithm to compute an inverse matrix: Given a square matrix A, we take the identity matrix I and write the system

$$[\, A \mid I \,].$$

Then we use the three row operations to reduce the left hand side to the identity matrix. If this is possible, we end up with

$$[\, I \mid B \,] = [\, I \mid A^{-1} \,].$$

If we cannot reduce the left hand side to the identity matrix, then the original matrix A is not invertible.

Example 1

Find the inverse of $A = \begin{bmatrix} 3 & 7 \\ 1 & 3 \end{bmatrix}$. We write

$$\left[\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 3 & 7 & 1 & 0 \end{array}\right] \xrightarrow{R2-3R1} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{-0.5R2} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & -1/2 & 3/2 \end{array}\right] \xrightarrow{R1-3R2} \left[\begin{array}{cc|cc} 1 & 0 & 3/2 & -7/2 \\ 0 & 1 & -1/2 & 3/2 \end{array}\right]$$

Hence, A is invertible and its inverse is $A^{-1} = \begin{bmatrix} 3/2 & -7/2 \\ -1/2 & 3/2 \end{bmatrix}$.

Example 2

Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. We write

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R1-2R2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

Because we have a row of zeros on the left hand side, we cannot transform the left hand side into the identity matrix by elementary row operations. Therefore, the matrix A is not invertible.

Example 3

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 0 & 2 & 4 \end{bmatrix}$. We write

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R3-2R2]{-R2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & -2 & 1 \end{array}\right] \xrightarrow{0.5R3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1/2 \end{array}\right]$$

$$\xrightarrow[R2-R3]{R1-3R3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 3 & -3/2 \\ 0 & 1 & 0 & -2 & 2 & -1/2 \\ 0 & 0 & 1 & 1 & -1 & 1/2 \end{array}\right] \xrightarrow{R1-2R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1/2 \\ 0 & 1 & 0 & -2 & 2 & -1/2 \\ 0 & 0 & 1 & 1 & -1 & 1/2 \end{array}\right]$$

Hence, the matrix A is invertible and the inverse is given by the right side of the last step above.

Example 4

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$. We write

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R3-2R2]{-R2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & -2 & 1 \end{array}\right]$$

Since the left half of the above matrix has a row of zeros, we cannot transform it into the identity matrix, using row operations. Therefore, A is NOT invertible.

When is a matrix invertible?

Sometimes we are not interested in the exact inverse of a matrix but only in the question of whether or not the matrix is invertible. Let's go back to the case of 2×2-matrices. When is the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

invertible? We have

$$\begin{aligned}[A|I] &= \left[\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right] \\ &= \left[\begin{array}{cc|cc} a_{11}a_{21} & a_{12}a_{21} & a_{21} & 0 \\ a_{11}a_{21} & a_{11}a_{22} & 0 & a_{11} \end{array}\right] \begin{array}{l} R1\times a_{21} \\ R2\times a_{11} \end{array} \\ &= \left[\begin{array}{cc|cc} a_{11}a_{21} & a_{12}a_{21} & a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} & -a_{21} & a_{11} \end{array}\right] \; R2-R1\end{aligned}$$

If $a_{11}a_{22} - a_{12}a_{21} = 0$, then A cannot be invertible. Since the expression $a_{11}a_{22} - a_{12}a_{21}$ is so important, we give it a special name.

Definition and Result: For a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the *determinant* of A is given by

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

If $\det(A) \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

If $\det(A) = 0$ then A is not invertible.

Example 1, revisited

The determinant of the matrix $A = \begin{bmatrix} 3 & 7 \\ 1 & 3 \end{bmatrix}$ is

$$\det(A) = 3 \cdot 3 - 1 \cdot 7 = 9 - 7 = 2.$$

The inverse according to our new formula is therefore $A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -7 \\ -1 & 3 \end{bmatrix}$, just as we had computed earlier.

Example 2, revisited

The determinant of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is

$$\det(A) = 1 \cdot 4 - 2 \cdot 2 = 4 - 4 = 0.$$

Hence, the matrix is not invertible, as we have already seen before.

Our theory so far only applies to matrices of size 2×2, therefore we cannot revisit the examples 3 and 4. We'll do that later. For now, we look at some more examples of matrices of size 2×2.

Example 5

The determinant of the matrix $A = \begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}$ is

$$\det(A) = 3 \cdot 2 - 1 \cdot 7 = 6 - 7 = -1.$$

Therefore, the inverse of A exists and is given by $A^{-1} = -\begin{bmatrix} 2 & -7 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$.

Example 6

The determinant of the matrix $A = \begin{bmatrix} 3 & -1 \\ 2 & 1/2 \end{bmatrix}$ is

$$\det(A) = 3 \cdot 0.5 - 2 \cdot (-1) = 3/2 + 2 = 7/2.$$

Therefore, the inverse of A exists and is given by $A^{-1} = \frac{2}{7}\begin{bmatrix} 1/2 & 1 \\ -2 & 3 \end{bmatrix}$.

Example 7

The determinant of the matrix $A = \begin{bmatrix} -7 & 2 \\ 4 & -8/7 \end{bmatrix}$ is

$$\det(A) = (-7) \cdot (-8/7) - 2 \cdot 4 = 8 - 8 = 0.$$

Hence, the matrix is not invertible.

Determinants for matrices of size 3×3

Determinants can be defined for square matrices of all sizes, and it is always true that if $\det(A) \neq 0$ then the matrix A is invertible. We will only consider the case of 3×3-matrices here, since it has a fairly simple form. Determinants of bigger matrices can be computed, but it takes time.

The determinant of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is given by

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

Unfortunately, there is no really simple formula for the inverse of a 3×3 matrix (or larger) analogous to the 2×2 case. To find the inverse, we still have to use the row-reduction algorithm.

This determinant formula can be remembered more easily by attaching the first two columns of the matrix A as columns 4 and 5 of a larger matrix and then taking products along the diagonal down with plus signs and products along the diagonal up with minus signs, i.e.,

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Example 3, revisited

The determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 0 & 2 & 4 \end{bmatrix}$ is

$$\det(A) = 1 \cdot 3 \cdot 4 + 2 \cdot 4 \cdot 0 + 3 \cdot 1 \cdot 2 - 0 \cdot 3 \cdot 3 - 2 \cdot 4 \cdot 1 - 4 \cdot 1 \cdot 2 = 18 - 14 = 4.$$

Hence, the matrix is invertible, as we already know.

Example 4, revisited

The determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ is

$$\det(A) = 1 \cdot 3 \cdot 2 + 2 \cdot 4 \cdot 0 + 3 \cdot 1 \cdot 2 - 0 \cdot 3 \cdot 3 - 2 \cdot 4 \cdot 1 - 2 \cdot 1 \cdot 2 = 12 - 12 = 0.$$

Hence, the matrix is not invertible, confirming our previous result.

Important Observation: If A is an invertible square matrix then the unique solution of the system of linear equations

$$Ax = b$$

is given by

$$x = A^{-1}b.$$

Example 1, revisited

The solution of the system

$$\begin{cases} 3x_1 + 7x_2 & = 4 \\ x_1 + 3x_2 & = 6 \end{cases}$$

is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -7 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -15 \\ 7 \end{bmatrix}$$

Practice Problems

1. Compute the determinant and find the inverse if it exists.

(a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ (d) $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ (f) $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

2. Solve the linear system of equations

For each of the matrices above solve the system $Ax = b$ where $b = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ in cases (a)-(d) and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in cases (e) and (f).

Solutions to Practice Problems

1.

(a) $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ (b) $A^{-1} = \begin{bmatrix} 1/4 & 0 \\ 0 & -1 \end{bmatrix}$ (c) $A^{-1} = \begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix}$

(d) and (f) are not invertible

$$\text{(e)} \quad A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

2.

(a) $x = \frac{1}{3}\begin{bmatrix} -1 \\ -4 \end{bmatrix}$, (b) $x = \begin{bmatrix} 1/4 \\ 2 \end{bmatrix}$, (c) $x = \begin{bmatrix} 7/3 \\ -2/3 \end{bmatrix}$,

In case (d) the solution is $\left\{ \begin{bmatrix} (1+2t)/3 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$.

$$\text{(e)} \quad x = \begin{bmatrix} -15 \\ 5 \\ 2 \end{bmatrix}.$$

In case (f), there is no solution.
NOTE: Even if the matrix is not invertible, there can still be a solution (see (d)), but it is not unique in that case.

Linear Algebra IV: Eigenvalues and Eigenvectors

Observation and introductory example

Consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so that $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$. For example

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad A \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

On the other hand, we also have

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

In these two cases, the vectors $A \begin{bmatrix} x \\ y \end{bmatrix}$ are simply multiples of the vectors $\begin{bmatrix} x \\ y \end{bmatrix}$, whereas in the examples above, there was no obvious relationship between $A \begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$. The goal of this section is to find and study these special cases where a vector multiplied by a matrix is simply a multiple of the vector.

> **Definition:** Let A be a square matrix. A vector v that is not the zero vector and a number λ are called *eigenvector* and *eigenvalue,* respectively if
>
> $$Av = \lambda v.$$

Example 1

The vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$, since

$$\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Example 2

The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 0$, since

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

NOTE: $\lambda = 0$ is allowed, but the eigenvector cannot be the zero vector.

Example 3

The vector $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 1$ since

$$\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

How to compute eigenvalues and eigenvectors?

We are looking for a nonzero solution of the equation $Av = \lambda v$. So let's rearrange:

$$\begin{aligned} Av - \lambda v &= 0 \\ (A - \lambda)v &= 0 \end{aligned}$$

except that $A - \lambda$ is meaningless, because A is a matrix and λ is a number. (The expression $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 5$ makes no sense.)

How can we deal with this? Answer: the identity matrix. Since $(\lambda I)v = \lambda(Iv) = \lambda v$, it's no problem to insert the identity matrix into our equation to make things balance. Thus

$$\begin{aligned} Av - \lambda v &= 0 \\ Av - \lambda I v &= 0 \\ (A - \lambda I)v &= 0. \end{aligned}$$

Now, the matrix $(A - \lambda I)$ is either invertible or it's not. If it's invertible, then

$$v = (A - \lambda)^{-1} 0 = 0.$$

But we want nonzero solutions, so this is useless. That means the only way to get nonzero solutions is if the matrix $(A - \lambda)$ is not invertible. That is,

$$\det(A - \lambda) = 0.$$

Result: The number λ is an eigenvalue of the square matrix A if and only if it satisfies the equation

$$\det(A - \lambda I) = 0.$$

If λ is an eigenvalue of the square matrix A then we find the corresponding eigenvector(s) by solving the linear system of equations

$$(A - \lambda I)v = 0.$$

Note: Any scalar multiple of an eigenvector is again an eigenvector.

Example 1

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

First we form the matrix

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}.$$

Then we calculate its determinant as

$$\det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1.$$

Setting the determinant to zero, we get the two eigenvalues $\lambda = \pm 1$. For $\lambda_1 = 1$ we solve the system

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The second row is satisfied, the first row reads $-x_1 + x_2 = 0$. Hence, $x_2 = t$ is a free variable, and $x_1 = t$ is the resulting solution, so that $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$ are the eigenvectors corresponding to $\lambda_1 = 1$ for any $t \neq 0$.

For $\lambda_2 = -1$ we solve the system

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The second row is satisfied, the first row reads $x_1 + x_2 = 0$. Hence, $x_2 = t$ is a free variable, and $x_1 = -t$ is the resulting solution, so that $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} s$ are the eigenvectors corresponding to $\lambda_2 = -1$ for any $s \neq 0$.

Example 2

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

First we form the matrix

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix}.$$

Then we calculate its determinant as

$$\det(A - \lambda I) = -\lambda(2-\lambda)(1-\lambda) - (2-\lambda) = (2-\lambda)(\lambda^2 - 1).$$

(Be careful to factor, not expand.)
Hence the eigenvalues are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$. For $\lambda_1 = 1$, we solve the system (writing only the coefficient matrix and suppressing the right hand column of zeros)

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have $x_3 = t$ as the free variable and solve for $x_2 = -2t$, and $x_1 = t$ to get the eigenvector $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -1$, we solve the system (writing only the coefficient matrix and suppressing the right hand column of zeros)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The second row gives the equation $3x_2 = 0$, which means $x_2 = 0$. We have $x_3 = t$ the free variable and solve for $x_1 = -t$ to get the eigenvector $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda_3 = 2$, we solve the system (writing only the coefficient matrix and suppressing the right hand column of zeros)

$$\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

The second row gives the equation $0x_2 = 0$, which means $x_2 = t$ is free. The first equation says $-2x_1 + x_3 = 0$, while the third equation says $x_1 - 2x_3 = 0$. The only way to make both of them true is to set $x_1 = x_3 = 0$. Hence we get the eigenvector $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Example 3

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

First we form the matrix

$$A - \lambda I = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix}.$$

Then we calculate its determinant as

$$\det(A - \lambda I) = -\lambda(2-\lambda)(3-\lambda) - (2-\lambda)(-2) = (2-\lambda)(\lambda-2)(\lambda-1).$$

Hence the eigenvalues are $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 1$. For the double eigenvalue $\lambda_1 = \lambda_2 = 2$, we solve the system

$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The first equation is $x_1 + x_3 = 0$. The variables $x_2 = s$ and $x_3 = t$ are free. The solution is $x_1 = -x_3 = -t$. Hence, we get essentially two eigenvectors. For one, we set $t = 0$ and $s = 1$. For the other, we set $t = 1$ and $s = 0$. Then

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda_3 = 1$, we solve the system (suppressing the column of zeros)

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case, we get $x_3 = t$ to be the free variable. The resulting solution then is $x_2 = x_3 = t$ and $x_1 = -2x_3 = -2t$. The eigenvector therefore is $v_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

Example 4

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

First we form the matrix

$$A - \lambda I = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & -11 \\ 1 & 1-\lambda \end{bmatrix}.$$

Then we calculate its determinant as

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2.$$

Setting the determinant to zero, we get

$$\lambda = \frac{1}{2}\left(2 \pm \sqrt{4-8}\right) = 1 \pm \sqrt{-1} = 1 \pm \mathrm{i}.$$

Hence, we get complex numbers as eigenvalues. Just to be sure, we check that $\lambda_1 = 1 + \mathrm{i}$ solves the original equation:

$$(1+\mathrm{i})^2 - 2(1+\mathrm{i}) + 2 = 1 + 2\mathrm{i} - 1 - 2 - 2\mathrm{i} + 2 = 0.$$

We can compute the corresponding eigenvectors in the same way as above, but the calculation involves complex numbers. For $\lambda_1 = 1 + \mathrm{i}$ we get

$$\begin{bmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{bmatrix} \longrightarrow (-\mathrm{i} \times R1 + R2) \longrightarrow \begin{bmatrix} -\mathrm{i} & -1 \\ 0 & 0 \end{bmatrix}.$$

Then $x_2 = t$ is a free variable and $x_1 = \mathrm{i}t$, so that the eigenvector is $v = \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix}$.
For $\lambda_2 = 1 - \mathrm{i}$ we get

$$\begin{bmatrix} \mathrm{i} & -1 \\ 1 & \mathrm{i} \end{bmatrix} \longrightarrow (\mathrm{i} \times R1 + R2) \longrightarrow \begin{bmatrix} \mathrm{i} & -1 \\ 0 & 0 \end{bmatrix}.$$

Then $x_2 = t$ is a free variable and $x_1 = -\mathrm{i}t$, so that the eigenvector is $v = \begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix}$.

Practice Problems

Find the eigenvalues and eigenvectors of the following matrices.

$$(a)\quad A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 3 & 1 \\ 2 & 0 & 0 \end{bmatrix}, \qquad (b)\quad A = \begin{bmatrix} -5 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix},$$

$$(c)\quad A = \begin{bmatrix} 3 & 2 & -6 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix}, \qquad (d)\quad A = \begin{bmatrix} 4 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & 6 & -3 \end{bmatrix},$$

$$(e)\quad A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}, \qquad (f)\quad A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}, \qquad (g)\quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$(h)\quad A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad (i)\quad A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

Solutions

For problem (a):

$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & -2 \\ -1 & \lambda-3 & -1 \\ -2 & 0 & \lambda \end{bmatrix} = \lambda^2(\lambda-3) - 4(\lambda-3) = (\lambda-3)(\lambda^2-4) = 0.$

So the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = -2$. To find the eigenvectors corresponding to an eigenvalue λ, we have to solve the system $(\lambda I - A)x = 0$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

For $\lambda_1 = 3$, we have to solve

$$\begin{aligned} 3x_1 - 2x_3 &= 0 \\ -x_1 - x_3 &= 0 \\ -2x_1 + 3x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 0 & -2 & 0 \\ -1 & 0 & -1 & 0 \\ -2 & 0 & 3 & 0 \end{array}\right] \overset{3R2+R1}{\underset{\longrightarrow}{3R3+2R1}} \left[\begin{array}{ccc|c} 3 & 0 & -2 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right].$$

Hence, $x_3 = 0$ and from the first equation also $x_1 = 0$. The eigenvectors are $\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, with $t \in \mathbb{R}$.

For $\lambda_2 = 2$,

$$\begin{aligned} 2x_1 - 2x_3 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \\ -2x_1 + 2x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ -1 & -1 & -1 & 0 \\ -2 & 0 & 2 & 0 \end{array}\right] \overset{2R2+R1}{\underset{\longrightarrow}{R3+R1}} \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Then $x_3 = t$ is free, $x_2 = -2t$ and $x_1 = t$. The eigenvectors are $\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ with $t \in \mathbb{R}$.

For $\lambda_3 = -2$,

$$\begin{aligned} -2x_1 - 2x_3 &= 0 \\ -x_1 - 5x_2 - x_3 &= 0 \\ -2x_1 - 2x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ -1 & -5 & -1 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \overset{2R2+R1}{\underset{R3+R1}{\longrightarrow}} \left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Then $x_3 = t$ is free, $x_1 = -t$ and $x_2 = 0$. The eigenvectors are $\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ with $t \in \mathbb{R}$.

For problem (b):
We have

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -5-\lambda & 2 & -3 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & 4-\lambda \end{bmatrix} \\ &= (-5-\lambda)(1-\lambda)(4-\lambda) - (1)(-2)(-5-\lambda) = (-5-\lambda)\left[(1-\lambda)(4-\lambda)+2\right] \\ &= -(5+\lambda)(\lambda^2 - 5\lambda + 6) = -(5+\lambda)(\lambda-2)(\lambda-3). \end{aligned}$$

Thus, the eigenvalues of A are $\lambda_1 = -5$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

The eigenvector v_1 associated with the eigenvalue $\lambda_1 = -5$ is the solution of the system

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} 0 & 2 & -3 \\ 0 & 6 & -2 \\ 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 2 & -3 & 0 \\ 0 & 6 & -2 & 0 \\ 0 & 1 & 9 & 0 \end{array}\right].$$

$R_1 - 2R_3 \to R_1$ and $R_2 - 6R_3 \to R_2$ gives

$$\left[\begin{array}{ccc|c} 0 & 0 & -21 & 0 \\ 0 & 0 & -56 & 0 \\ 0 & 1 & 9 & 0 \end{array}\right].$$

$(-1/21)R_1 \to R_1$ gives

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -56 & 0 \\ 0 & 1 & 9 & 0 \end{array}\right].$$

$R_2 + 56R_1 \to R_2$ and $R_3 - 9R_1 \to R_3$ gives

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

Finally, $R_3 \leftrightarrow R_1$ and $R_2 \leftrightarrow R_3$ gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The solution is $y = z = 0$ and $x = t$. Thus, the eigenvectors are $v_1 = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ where t is any real number except 0.

Next, for $\lambda_2 = 2$ we have

$$(A - \lambda_2 I)v_2 = \begin{pmatrix} -7 & 2 & -3 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus

$$\left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right].$$

$R_3 + R_2 \to R_3$ and $R_1 + 2R_2 \to R_1$ gives

$$\left[\begin{array}{ccc|c} -7 & 0 & -7 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$(-1/7)R_1 \to R_1$ and $-R_2 \to R_2$ gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The solution is thus $x = -z$ and $y = -2z$. With $z = s$, we get $v_2 = s\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ where s is any real number except 0.

Finally, for $\lambda_3 = 3$ we have

$$(A - \lambda_3 I)v_3 = \begin{pmatrix} -8 & 2 & -3 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} -8 & 2 & -3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

$R_2 + 2R_3 \to R_2$ and $R_1 - 2R_3 \to R_1$ gives

$$\left[\begin{array}{ccc|c} -8 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

$(-1/8)R_1 \to R_1$ and $R_2 \leftrightarrow R_3$ gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 5/8 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The solution is $x = -5z/8$ and $y = -z$. With $z = r$, we get $v_3 = r \begin{bmatrix} -5/8 \\ -1 \\ 1 \end{bmatrix}$ where r is any real number except 0.

For problem (c):

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 2 & -6 \\ 0 & -2-\lambda & 1 \\ 0 & 4 & 1-\lambda \end{bmatrix} \\ &= (3-\lambda)(-2-\lambda)(1-\lambda) + 0 + 0 - 0 - 0 - (3-\lambda)(4) \\ &= (3-\lambda)[-2-\lambda+\lambda^2-4] \\ &= (3-\lambda)[\lambda^2-\lambda-6] \\ &= (3-\lambda)(\lambda+3)(\lambda-2). \end{aligned}$$

Thus $\lambda_1 = 3, \lambda_2 = -3, \lambda_3 = 2$.

Eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 4/3 \\ -1 \\ 1 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 11/2 \\ 1/4 \\ 1 \end{bmatrix}.$$

For problem (d):

$$\begin{aligned} \det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & 2 & -5 \\ 0 & 1-\lambda & 2 \\ 0 & 6 & -3-\lambda \end{pmatrix} \\ = (4-\lambda)\left[(1-\lambda)(-3-\lambda) - 12\right] \\ = (4-\lambda)(\lambda^2 + 2\,\lambda - 15) = (4-\lambda)(\lambda-3)(\lambda+5) \end{aligned}$$

Thus, the eigenvalues of A are $\lambda_1 = -5$, $\lambda_2 = 3$ and $\lambda_3 = 4$.

Eigenvectors are

$$v_1 = \begin{bmatrix} 17/27 \\ -1/3 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

For problem (e):
Eigenvalues are

$$\lambda_1 = 5, \lambda_2 = -1.$$

Eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

For problem (f):
Eigenvalues are

$$\lambda_1 = 5, \lambda_2 = -5.$$

Eigenvectors are

$$v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

For problem (g):
Eigenvalues are

$$\lambda_1 = -\mathrm{i}, \lambda_2 = \mathrm{i}.$$

Eigenvectors are

$$v_1 = \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix}.$$

For problem (h):
Eigenvalues are

$$\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = 4.$$

Eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 5/3 \\ 1 \\ 0 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

For problem (i):
Eigenvalues are

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2.$$

Eigenvectors are

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Functions of several variables I: Introduction

Introductory example

We can measure the rate of food uptake of a single individual as a function of temperature. We will probably find some optimal temperature T_{opt}, where the uptake rate is highest. At lower temperatures, it is too cold, at higher temperatures, it is too hot for the organism to function properly. If we denote the rate by r and temperature by T then we might try to model this situation with the function

$$r(T) = r_{\max} \exp(-(T - T_{\text{opt}})^2).$$

We can also measure the uptake rate at a constant temperature but in the presence of other individuals. Typically, we see the uptake rate decrease in the presence of others due to competition for food. We have seen such functions before; for example

$$r(N) = r_{\max} \frac{N}{1+N},$$

where N is the number of individuals around. Now we want to vary T and N independently. We could simply multiply the two expressions and get

$$r(N,T) = r_{\max} \frac{N}{1+N} \exp(-(T - T_{\text{opt}})^2).$$

This function now depends on the two variables T and N. While it is easy to plot the two functions of a single variable above, it is much harder to get a good impression of the function of two variables. Figure 5 shows how r depends on N and T individually, as well as together. Not only is it more difficult to visualize functions of two and more variables, it is also harder to analyze them. The goal of this chapter is to define concepts such as level sets and derivatives for these functions.

Definition: The set $\mathbb{R}^n$ is the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ where all x_i are real numbers. So $\mathbb{R}^1 = \mathbb{R}$, the real numbers; $\mathbb{R}^2$ is the set of points (x_1, x_2) in the plane; $\mathbb{R}^3$ are the points in space. We also use the notation (x, y) and (x, y, z) for points in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

A real-valued function on some subset $D \subset \mathbb{R}^n$ is a function $f \colon D \to \mathbb{R}$ that assigns a real number to each element in D. The set D is called the domain of definition of f.

The graph of a function $f \colon D \to \mathbb{R}$ of two variables is the set

$$G = \{(x, y, z) \in \mathbb{R}^3 \ : \ (x, y) \in D, z = f(x, y)\}.$$

In particular, the graph is a subset of three-dimensional space, and as such, it is not so easy to visualize. Graphs of functions of more variables are defined analogously, but as they are subsets of spaces of dimension 4 and higher, they cannot be visualized in a similar manner.

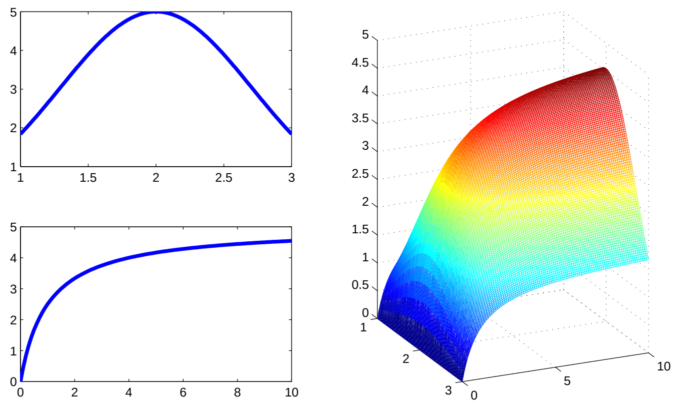

Figure 5: Uptake rate as a function of temperature alone (top left), as a function of population density alone (bottom left), and as a function of both independent variables (right).

Examples

1. $n = 2, D = \mathbb{R}^2, f(x_1, x_2) = x_1^2 + x_2^2$. Then for example

$$f(0,0) = 0, \quad f(1,0) = 1, \quad f(1,1) = 2, f(2,4) = 20.$$

If we fix $x_2 = 0$ the we have a function of a single variable $f(x_1, 0) = x_1^2$. Similarly, if we fix $x_1 = 0$, then $f(0, x_2) = x_2^2$. For a visualization of this function, see Figure 6.

2. $n = 2, D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1\}, f(x, y) = x + y$. See Figure 6.

3. Find the largest possible domain of the function $f(x, y) = \sqrt{y^2 - x}$. Answer: The square root is real only if $y^2 - x \ge 0$, i.e., $y^2 \ge x$. This gives two conditions:

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ y \ge \sqrt{x} \text{ or } y < -\sqrt{x}\}.$$

See Figure 7 for the domain and the graph of the function.

For functions of two variables, there is a simple way to visualize their behaviour. The idea is the same as in topographic maps where the contour lines indicate points of equal altitude.

Definition: The level set, L_c, or contour line of a function $f(x, y)$ is the set of all points $(x, y) \in D$ where f has a given value c; i.e.,

$$L_c = \{(x, y) \in D \ : \ f(x, y) = c\}.$$

Examples, revisited

1. $f(x, y) = x^2 + y^2$. Pick some value c. Then

$$f(x, y) = c \Leftrightarrow x^2 + y^2 = c \Leftrightarrow y = \pm\sqrt{c - x^2}.$$

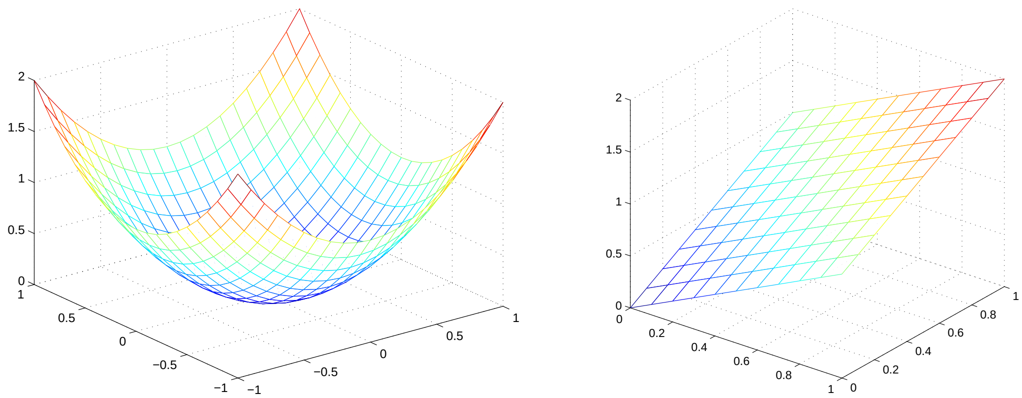

Figure 6: Left: The graph of the function $f(x_1, x_2) = x_1^2 + x_2^2$. Right: The graph of the function $f(x, y) = x + y$.

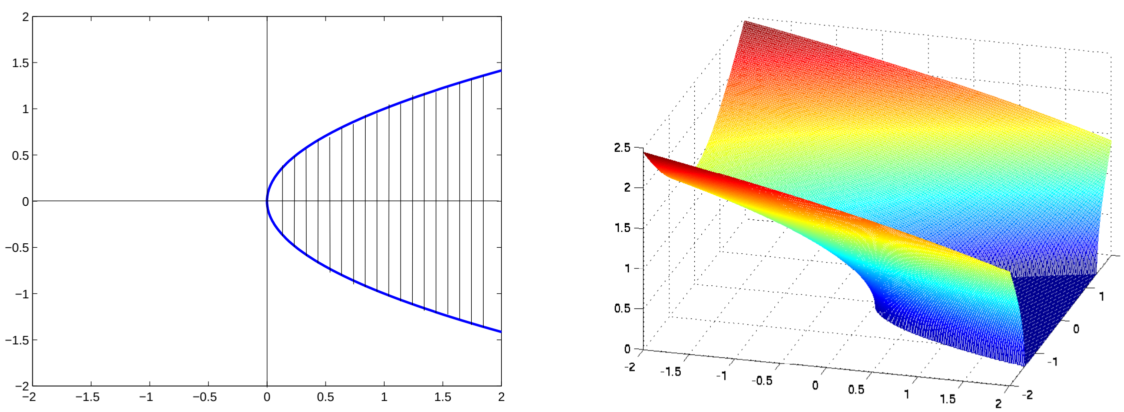

Figure 7: Left: The white area is the maximal domain of definition of the function $f(x, y) = \sqrt{y^2 - x}$. In the shaded area, the square root is not real. Right: The graph of the function.

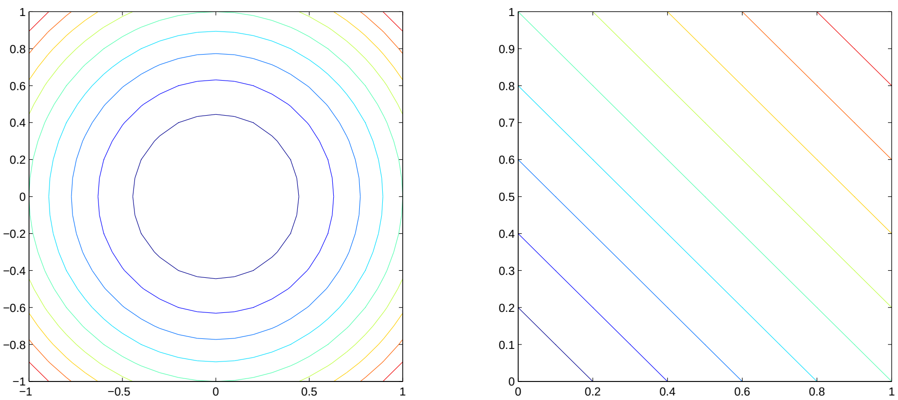

Figure 8: Left: Level sets of the function $f(x_1, x_2) = x_1^2 + x_2^2$. Right: Level sets of the function $f(x, y) = x + y$.

Hence, the level sets for $c > 0$ are concentric circles around the origin. Level sets for $c < 0$ are empty, see Figure 8.

2. $f(x, y) = x + y$. Again, we pick some value c. Then

$$f(x, y) = c \Leftrightarrow x + y = c \Leftrightarrow y = c - x.$$

Hence, the level sets are straight lines with slope -1, see Figure 8.

3. $f(x, y) = \sqrt{y^2 - x}$. Fix c. Then

$$f(x, y) = c \Leftrightarrow y^2 - x = c^2 \Leftrightarrow y = \pm\sqrt{x + c^2}.$$

Hence, the level sets look like the square root function, but they are shifted and flipped, see Figure 9.

We would now like to carry over all the concepts we know from functions of a single variable to functions of several variables, i.e., limits, continuity, derivatives, integrals. All this material is typically taught in a course on "multivariable calculus". It turns out that the general case is much more tricky than the case of a single variable, so much so that we will only cover a small fraction in this course. The following example about continuity shows how things can go "wrong"

Example on limits and continuity

Define the function

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad \text{on} \quad D = \mathbb{R}^2 \backslash \{0\}.$$

What is the limit of f as (x, y) approach the origin $(0, 0)$? Well, it depends on how one approaches the origin.

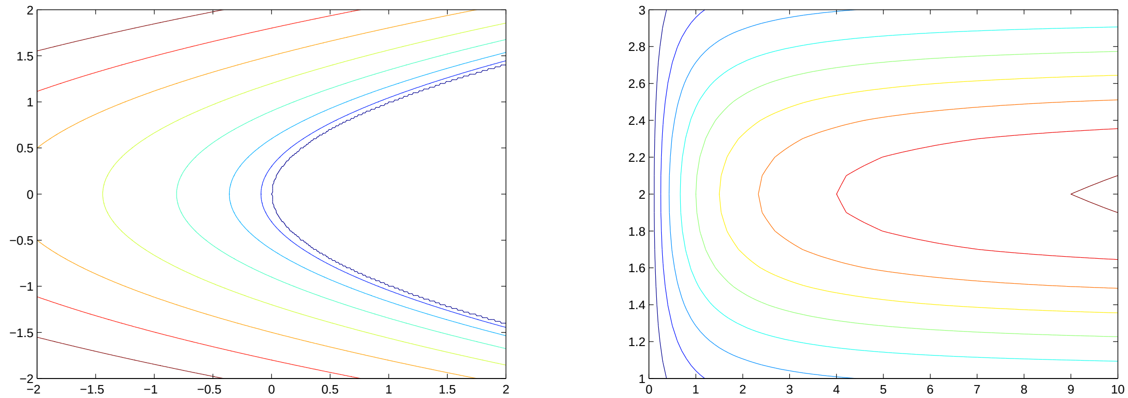

Figure 9: Left: Level sets of the function $f(x,y) = \sqrt{y^2 - x}$. Right: Level sets of the function from the introductory example: $r(N,T) = 5\frac{N}{1+N}\exp(-(T-2)^2)$.

1. At first, fix $y = 0$ and consider the function of a single variable

$$g(x) = f(x,0) = \frac{x^2 - y^2}{x^2 + y^2}|_{y=0} = \frac{x^2}{x^2} = 1.$$

   Hence, in the limit as $x \to 0$, we get one.

2. Next, fix $x = 0$ and consider the function of a single variable

$$h(y) = f(0,y) = \frac{x^2 - y^2}{x^2 + y^2}|_{x=0} = \frac{-y^2}{y^2} = -1.$$

   Hence, in the limit as $y \to 0$, we get minus one.

3. More generally, pick some $m \neq 0$ and choose the straight line $(x,y) = (x,mx)$ on which to approach the origin. Then in the limit

$$\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1-m}{1+m}.$$

   Hence, the value of the function is constant along such lines, and the limit is different for all the possible straight lines on which we approach the origin. See Figure 10 for a plot of this "weird" function. (Note that we have just computed contour sets for this function.)

Important Observation: We can always obtain functions of a single variable from a function of several variables by holding all but one of the variables constant, as we have done in a few examples above.

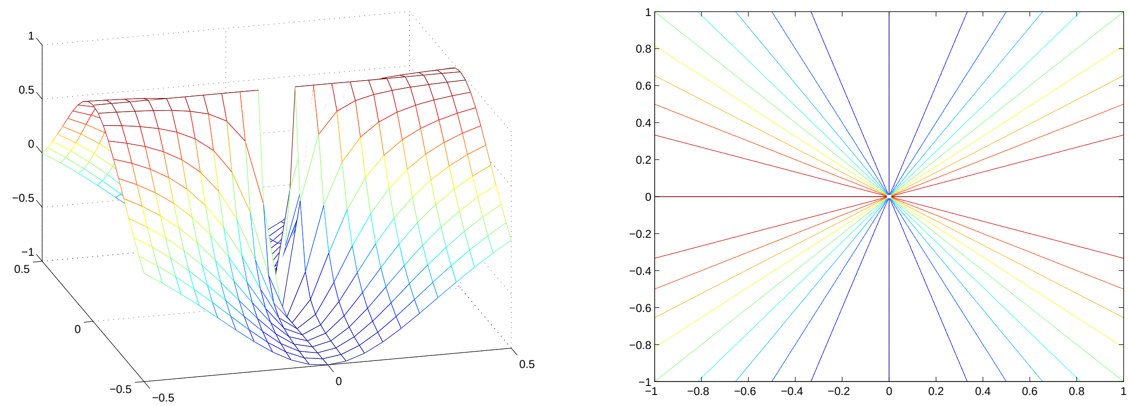

Figure 10: Left: Graph of the function $f(x,y) = (x^2 - y^2)/(x^2 + y^2)$. Right: Level sets of the same function.

Functions of several variables II: Partial derivatives

When we do experiments with several control variables, for example temperature and population density as in the previous section, then we should vary only one of the conditions at a time to see how the response of the system changes with respect to that particular variable.

This idea is precisely the idea behind the mathematical concept of partial derivative. We fix all the variables but one, we are left with a function of a single variable, and we know how to differentiate that.

Definition: Let $f(x_1, \ldots x_n)$ be a real-valued function of n variables. Then the *partial derivative* with respect to x_k is defined as

$$\frac{\partial f}{\partial x_k} = \lim_{h \to 0} \frac{1}{h}[f(x_1, \ldots, x_k + h, x_{k+1}, \ldots x_n) - f(x_1, \ldots, x_k, \ldots x_n).$$

Note: this is the same definition as for a single variable with all the other variables considered fixed pr parameters. Note also the different notation, the "curly" ∂ versus the straight d.

Examples

1. $f(x,y) = x^2 y$. To find the partial derivative with respect to x, let us fix y. Then we differentiate with respect to x and get

$$\frac{\partial f}{\partial x} = 2xy.$$

Conversely, if we fix x and differentiate with respect to y we get

$$\frac{\partial f}{\partial y} = x^2.$$

2. $f(x, y) = ye^{xy}$. Then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= y^2 e^{xy} && \text{(using the chain rule)} \\
\frac{\partial f}{\partial y} &= e^{xy} + yxe^{xy} = (1 + xy)e^{xy} && \text{(using the product rule).}
\end{aligned}$$

3. $f(x, y) = \frac{\sin(xy)}{x^2+1}$. Then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{y\cos(xy)(x^2+1) - 2x\sin(xy)}{(x^2+1)^2} && \text{(using the quotient rule)} \\
\frac{\partial f}{\partial y} &= \frac{x\cos(xy)}{x^2+1}.
\end{aligned}$$

4. $f(x, y, z) = e^{xy^2}(1 + z^2)$.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= y^2 e^{xy^2}(1 + z^2) && \text{(using the chain rule)} \\
\frac{\partial f}{\partial y} &= 2xye^{xy^2}(1 + z^2) && \text{(using the chain rule)} \\
\frac{\partial f}{\partial z} &= 2ze^{xy^2}.
\end{aligned}$$

Geometric interpretation of partial derivatives

Recall that, for a function of a single variable, $g(x)$, the derivative, $g'(x)$, gives the slope of the tangent line at a given point. In fact, the tangent line at a point $(x_0, y_0) = (x_0, g(x_0))$ can be written as

$$y - y_0 = g'(x_0)(x - x_0).$$

Now, for functions of two (or more) variables, the situation is very similar since we obtained partial derivatives from fixing all but one variables and considering essentially a function of a single variable. Therefore, given a function $f(x, y)$ and a point (x_0, y_0) we have $z_0 = f(x_0, y_0)$. Then we can define two functions, namely

$$g_1(x) = f(x, y_0), \qquad g_2(y) = f(x_0, y).$$

Then the partial derivatives $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ are the slopes of the tangent line at the point (x_0, y_0, z_0) to the curves

$$z = f(x, y_0), \qquad \text{and} \qquad z = f(x_0, y),$$

respectively. See Figure 11

In the case of a single variable, we know how to find the *tangent line* to the graph of a function at a point. In the case of two variables, we would like to find the *tangent plane* to the graph of the function, i.e., to the surface in space. This tangent plane necessarily contains the two

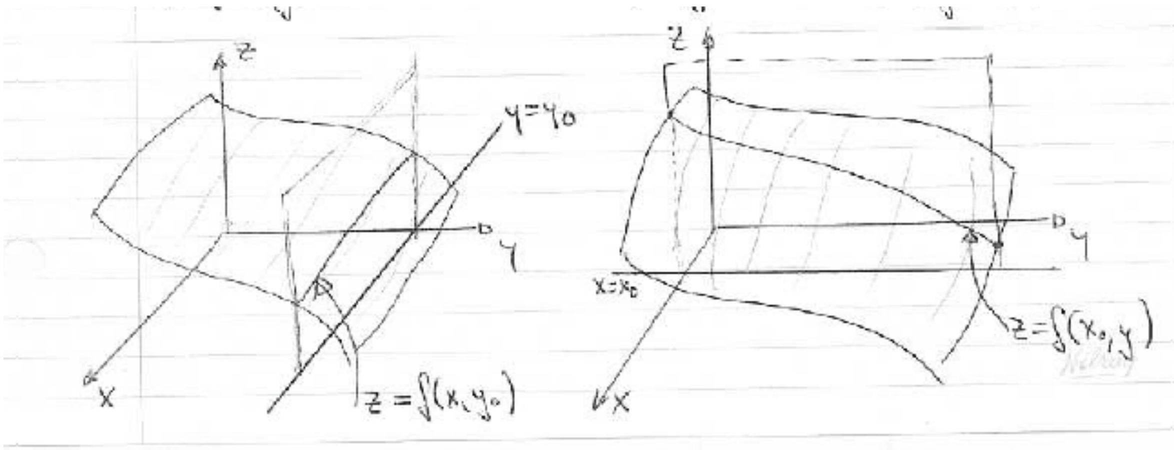

Figure 11: Two one-dimensional curves to which the partial derivatives give the slope.

tangent lines, whose slope we just calculated. Therefore, we can write down the equation of the tangent plane easily.

Fact: Let $f(x, y)$ be a real-valued function of two variables. If the tangent plane to the graph of f at the point $(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ exists, then it is given by the equation

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Examples

1. $f(x, y) = x^2 + y^2$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y.$$

For $(x_0, y_0) = (0, 0)$, we have $f(0, 0) = 0$ and hence the equation of the tangent plane is

$$z - 0 = 0(x - 0) + 0(y - 0), \quad \text{or} \quad z = 0.$$

Hence, the tangent plane is simply the x-y-plane.
At $(x_0, y_0) = (1, 0)$ we have $f(1, 0) = 1$ and hence the equation of the tangent plane there is

$$z - 1 = 2(x - 1) + 0(y - 0), \quad \text{or} \quad z = 2x - 1.$$

At $(x_0, y_0) = (1, 1)$ we have $f(1, 1) = 2$ and hence the equation of the tangent plane there is

$$z - 2 = 2(x - 1) + 2(y - 1), \quad \text{or} \quad z = 2x + 2y - 2.$$

See Figure 12 for a plot of the function and the tangent plane at $(1, 1, 2)$.

2. $f(x,y) = x + y$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = 1.$$

Hence, the tangent plane at (x_0, y_0, z_0) is

$$z - z_0 = x - x_0 + y - y_0 \quad \text{or} \quad z = x + y,$$

since $z_0 = x_0 + y_0$. Hence the tangent plane is exactly the same as the function, which is a plane. We know that the same is true for functions of a single variable, where the tangent line to a linear function is the function itself.

3. $f(x,y) = \sqrt{y^2 - x}$. The partial derivatives are

$$\frac{\partial f}{\partial x} = -\frac{1}{2\sqrt{y^2 - x}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - x}} \qquad \text{(using the chain rule).}$$

Then the general equation of the tangent plane is

$$z - z_0 = -\frac{1}{2\sqrt{y_0^2 - x_0}}(x - x_0) + \frac{y_0}{\sqrt{y_0^2 - x_0}}(y - y_0).$$

For example, at $(0, 1, 1)$ we get

$$z - 1 = -\frac{1}{2}(x - 0) + 1(y - 1) \quad \text{or} \quad z = -\frac{1}{2}x + y.$$

See Figure 12 for a plot of the function and the tangent plane at $(0, 1, 1)$.

Linear approximation

Again, we start by recalling the case of a single variable with a function $g(x)$. The tangent line at x_0 gives the linear approximation of the function near that point as

$$g(x) \approx g(x_0) + g'(x_0)(x - x_0),$$

which is also known as the first term in the Taylor series approximation. Now let's do the same thing for a function of two variables, $f(x,y)$, and use the tangent plane to approximate the function locally.

Definition: The linear approximation of a function $f(x,y)$ near a point (x_0, y_0) is given by

$$L(x,y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0),$$

provided the function is differentiable.

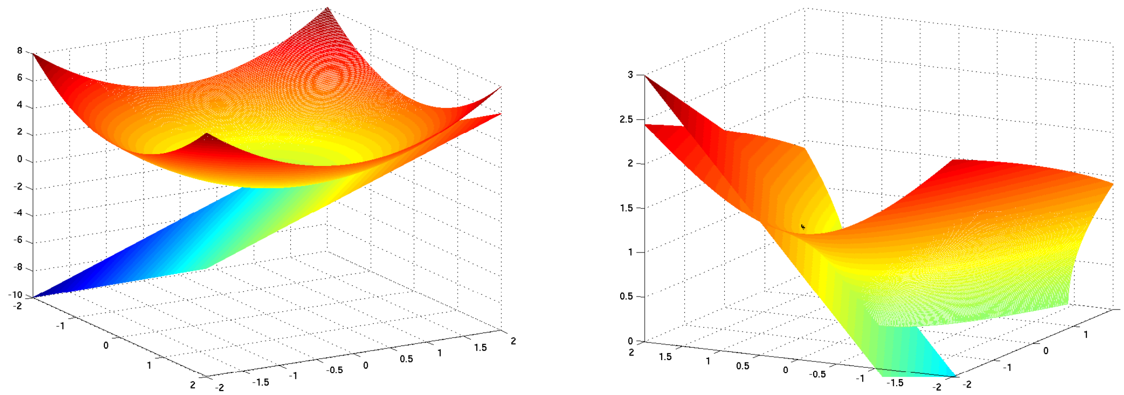

Figure 12: Left: Graph of the function $f(x, y) = x^2 + y^2$ and the tangent plane at the point $(1, 1, 2)$. Right: Graph of the function $f(x, y) = \sqrt{y^2 - x}$ and the tangent plane at the point $(0, 1, 1)$.

Examples

1. Find the linear approximation of $f(x, y) = x^2y + 2xe^y$ at the point $(2, 0)$.
   First, find the partial derivatives

$$\frac{\partial f}{\partial x} = 2xy + 2e^y, \quad \frac{\partial f}{\partial y} = x^2 + 2xe^y.$$

   Evaluate at the point $(2, 0)$ to get

$$\frac{\partial f}{\partial x} = 2, \quad \frac{\partial f}{\partial y} = 8.$$

   Then with $f(2, 0) = 4$ we get

$$L(x, y) = 4 + 2(x - 2) + 8(y - 0) = 2x + 8y.$$

   The linear approximation should, as the name implies, be a good approximation to the function, for points close to $(2, 0)$. So let's check the point $(2.1, 0.1)$. The original function has $f(2.1, 0.1) = 5.08$. The linear approximation has $L(2.1, 0.1) = 5$. So the linear approximation is pretty good, for points close to $(2, 0)$.

   If we try a point further away, then $f(3, 4) = 363.6$, while $L(x, y) = 38$. So, as you might expect, the linear approximation is not very good for points too far from $(2, 0)$.

2. Find the linear approximation of $f(x, y) = \ln(x - 2y^2)$ at the point $(3, 1)$.
   First, find the partial derivatives

$$\frac{\partial f}{\partial x} = \frac{1}{x - 2y^2}, \quad \frac{\partial f}{\partial y} = \frac{-4y}{x - 2y^2}.$$

Evaluate at the point $(3,1)$ to get

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = -4.$$

Then with $f(3,1) = \ln(1) = 0$ we get

$$L(x,y) = 0 + 1(x-3) - 4(y-1) = x - 4y + 1.$$

Let's check the point $(2.9, 0.9)$. We have $f(2.9, 0.9) = 0.247$ but $L(2.9, 0.9) = 0.3$. So this is in the ballpark.

Definition and notation: If $f(x,y)$ is a function of two variables, then the gradient of f is the row vector

$$\text{grad}(f) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right].$$

The linear approximation can then be written as

$$L(x,y) = f(x_0,y_0) + \text{grad}(f)\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}.$$

Functions of several variables III: Vector-valued functions

Motivation

We want to be able to describe several quantities that are all dependent on the same variables. Going back to Section 5.5 in the textbook, for example, we described a predator-prey system with prey b and predator p as

$$\begin{aligned} \frac{db}{dt} &= rb - cbp \\ \frac{dp}{dt} &= \gamma cbp - mp, \end{aligned}$$

with parameters r, c, γ, m. Hence, the growth rate of the prey and the predator both depend on the densities of prey and predator. We can write the right-hand side of the above system as a single, vector-valued function of the two variables b, p as follows:

$$F(b,p) = \begin{bmatrix} rb - cbp \\ \gamma cbp - mp \end{bmatrix}.$$

Definition: A vector-valued function F of the variables $x_1, \ldots x_n$ is a function of the form

$$F(x_1, \ldots x_n) = \begin{bmatrix} f_1(x_1, \ldots x_n) \\ f_2(x_1, \ldots x_n) \\ \vdots \\ f_k(x_1, \ldots x_n) \end{bmatrix},$$

where all the functions $f_j(x_1, \ldots x_n)$ are real-valued functions.

Examples

1. The following is a function of two variables that produces a vector of length 2:

$$F(x,y) = \begin{bmatrix} 3x^2 - 2y \\ x^3 \sin(y) \end{bmatrix}.$$

For every point (x, y) that we put in, we obtain a 2×1-vector. For example,

$$F(1,0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad F(2,\pi) = \begin{bmatrix} 12 - 2\pi \\ 0 \end{bmatrix}, \quad F(1,\pi/2) = \begin{bmatrix} 3 - \pi \\ 1 \end{bmatrix}.$$

2. The right-hand side of Newton's law of cooling

$$\begin{aligned} \frac{dH}{dt} &= \alpha(A - H) \\ \frac{dA}{dt} &= \beta(H - A) \end{aligned}$$

(see Section 5.5 in the book), can be written as a vector-valued function

$$F(H,A) = \begin{bmatrix} \alpha(A - H) \\ \beta(H - A) \end{bmatrix}.$$

3. The equation of two competing populations, a, b

$$\begin{aligned} \frac{da}{dt} &= ra(1 - a - \eta b) \\ \frac{db}{dt} &= sb(1 - b - \mu a) \end{aligned}$$

(see Section 5.5 in the book), can be written as a vector-valued function

$$F(a,b) = \begin{bmatrix} ra(1 - a - \eta b) \\ sb(1 - b - \mu a) \end{bmatrix}.$$

Linear approximation and the Jacobian matrix

Since we will be only dealing with differential equations of two variables in this course, we only consider the case of two variables and two equations, or equivalently, vector-valued functions with two components, i.e.,

$$F(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}.$$

From the previous section, we know how to find the linear approximation for each of the two functions, $f(x, y)$ and $g(x, y)$, namely

$$f(x,y) \approx f(x_0,y_0) + \text{grad}(f) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix},$$

for f and

$$g(x,y) \approx g(x_0,y_0) + \text{grad}(g) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix},$$

for g, respectively. Now we put these two linear approximations together to obtain the linear approximation for $F(x,y)$.

$$\begin{aligned} F(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} &\approx \begin{bmatrix} f(x_0,y_0) + \text{grad}(f) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \\ g(x_0,y_0) + \text{grad}(g) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} f(x_0,y_)) \\ g(x_0,y_0) \end{bmatrix} + \begin{bmatrix} \text{grad}(f) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \\ \text{grad}(g) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \end{bmatrix} \\ &= F(x_0,y_0) + \begin{bmatrix} \text{grad}(f) \\ \text{grad}(g) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \equiv L(x,y). \end{aligned}$$

Definition and notation: The matrix of partial derivatives

$$J(x,y) = \begin{bmatrix} \text{grad}(f) \\ \\ \text{grad}(g) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix}$$

is called the *Jacobian matrix* of the function $F(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$. Using this matrix, we can write the linear approximation to $F(x,y)$ as

$$L(x,y) = F(x_0,y_0) + J(x,y) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}.$$

Example 1

Let $F(x,y) = \begin{bmatrix} x^2y - y^3 \\ 2x^3y^2 + y \end{bmatrix}$. Find the Jacobian matrix, evaluate it at $(1,2)$ and find the linear approximation at that point.

Solution:

$$J(x,y) = \begin{bmatrix} 2xy & x^2-3y^2 \\ 6x^2y^2 & 4x^3y+1 \end{bmatrix}, \qquad J(1,2) = \begin{bmatrix} 4 & -11 \\ 24 & 9 \end{bmatrix}.$$

With $F(1,2) = \begin{bmatrix} -6 \\ 10 \end{bmatrix}$, we get the linear approximation

$$L(x,y) = \begin{bmatrix} -6 \\ 10 \end{bmatrix} + \begin{bmatrix} 4 & -11 \\ 24 & 9 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} = \begin{bmatrix} 4x-11y+12 \\ 24x+9y-32 \end{bmatrix}.$$

Let's check the point $(1.1, 1.9)$. $F(1.1, 1.9) = (-4.56, 11.5098)$ while $L(1.1, 1.9) = (-4.5, 11.5)$. So this is a pretty close approximation.

Example 2

Let $F(x, y) = \begin{bmatrix} ye^{-x} \\ \sin(x) + \cos(y) \end{bmatrix}$. Find the Jacobian matrix, evaluate it at $(0, 0)$ and find the linear approximation at that point.

Solution:

$$J(x, y) = \begin{bmatrix} -ye^{-x} & e^{-x} \\ \cos(x) & -\sin(y) \end{bmatrix}, \qquad J(1, 2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

With $F(0, 0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we get the linear approximation

$$L(x, y) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x - 0 \\ y - 0 \end{bmatrix} = \begin{bmatrix} y \\ x + 1 \end{bmatrix}.$$

Let's check the point $(-0.1, 0.1)$. $F(-0.1, 0.1) = (0.1105, 0.895)$ while $L(-0.1, 0.1) = (0.1, 0.9)$. So this is a close approximation as well.

Example 3

Let $F(x, y) = \begin{bmatrix} \sqrt{2x + y} \\ x - y^2 \end{bmatrix}$. Find the Jacobian matrix, evaluate it at $(1, 2)$ and find the linear approximation at that point.

Solution:

$$J(x, y) = \begin{bmatrix} \frac{1}{\sqrt{2x+y}} & \frac{1}{2\sqrt{2x+y}} \\ 1 & -2y \end{bmatrix}, \qquad J(1, 2) = \begin{bmatrix} 1/2 & 1/4 \\ 1 & -4 \end{bmatrix}.$$

With $F(0, 0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ we get the linear approximation

$$L(x, y) = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 1/2 & 1/4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix} = \begin{bmatrix} x/2 + y/4 + 1 \\ x - 4y + 4 \end{bmatrix}.$$

Let's check the point $(0.9, 2.1)$. $F(0.9, 2.1) = (1.9748, -3.51)$ while $L(x, y) = (1.975, -3.5)$. So this is a very good approximation indeed.

Systems of differential equations I: the linear case

Recall that a single linear differential equation

$$\frac{d}{dt}x(t) = ax(t), \qquad x(0) = x_0$$

has the solution $x(t) = e^{at}x_0$. In particular, if $a > 0$, the solution grows to infinity, and if $a < 0$ then the solution approaches zero. We now want to generalize this result to two coupled equations

$$\begin{aligned} \frac{d}{dt}x_1(t) &= ax_1(t) + bx_2(t) & \qquad x_1(0) &= x_{10} \\ \frac{d}{dt}x_2(t) &= rx_1(t) + sx_2(t) & \qquad x_2(0) &= x_{20}. \end{aligned}$$

We can also write this in matrix notation (suppressing the argument t for the moment) as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ r & s \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \equiv A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ r & s \end{bmatrix}.$$

Observation I: Eigenvalues and eigenvectors provide solutions

Suppose we are looking for solutions of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Plugging this expression into the equation above, we get

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \lambda e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = e^{\lambda t}A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Canceling the exponential term on both sides, we find the condition

$$\lambda\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

which means that λ has to be an eigenvalue of A and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ has to be the corresponding eigenvector.

Fact: If λ is an eigenvalue of A and v is the corresponding eigenvector, then

$$x(t) = e^{\lambda t}v$$

is a solution of the linear system of differential equations

$$\frac{d}{dt}x(t) = Ax(t),$$

where $x = (x_1, \ldots x_n)^T$, and A is an $n \times n$-matrix.

Example 1

Take the system

$$\frac{d}{dt}x_1(t) = x_1(t) + 4x_2(t)$$
$$\frac{d}{dt}x_2(t) = 2x_1(t) - x_2(t),$$

with matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$. The eigenvalues of A are given by the equation

$$(1-\lambda)(-1-\lambda) - 8 = \lambda^2 - 9 = 0.$$

Hence, the eigenvalues are $\lambda = 3$ and $\mu = -3$. The corresponding eigenvectors are

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

respectively. Hence, we have the two solutions

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\lambda t} v = e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\mu t} w = e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have two solutions, but we don't have any constants in the equation yet with which we could match the initial condition. This is the next topic.

Observation II: Sums and multiples of solutions are solutions

Suppose we have two solutions $w(t)$ and $z(t)$ for the same system, i.e.,

$$\frac{d}{dt}w(t) = Aw(t), \qquad \frac{d}{dt}z(t) = Az(t).$$

Now pick two numbers C_1, C_2 and form

$$x(t) = C_1 w(t) + C_2 z(t).$$

Then $x(t)$ is also a solution since

$$\frac{d}{dt}x(t) = C_1 \frac{d}{dt}w(t) + C_2 \frac{d}{dt}z(t) = C_1 Aw(t) + C_2 Az(t) = A[C_1 w(t) + C_2 z(t)] = Ax(t).$$

Example 1, continued

Find the solution of the system

$$\frac{d}{dt}x_1(t) = x_1(t) + 4x_2(t)$$
$$\frac{d}{dt}x_2(t) = 2x_1(t) - x_2(t),$$

with initial values $x_{10} = 3, x_{20} = 3$.
We already know the eigenvalues and eigenvectors, so we get the general solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Substituting the initial conditions, i.e., setting $t = 0$, we get

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence, we solve the system

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & -1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 3 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right].$$

The solution is $C_1 = 2, C_2 = -1$, and therefore the solution to the differential equation is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Or, equivalently

$$\begin{aligned} x_1(t) &= 4e^{3t} - e^{-3t} \\ x_2(t) &= 2e^{3t} + e^{-3t}. \end{aligned}$$

We can always check our solution by differentiating:

$$\frac{d}{dt}x_1(t) = 12e^{3t} + 3e^{-3t}, \quad x_1 + 4x_2 = 4e^{3t} - e^{-3t} + 8e^{3t} + 4e^{-3t} = 12e^{3t} + 3e^{-3t}.$$

The two expressions agree. Similar for x_2.

Example 2

Find the solution of Newton's law of cooling

$$\begin{aligned} \frac{d}{dt}x_1(t) &= 3[x_2(t) - x_1(t)] & \qquad x_{10} &= 5 \\ \frac{d}{dt}x_2(t) &= x_1(t) - x_2(t) & \qquad x_{20} &= 1. \end{aligned}$$

First, we observe that the eigenvalues of the matrix $A = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$ are $\lambda = -4, \mu = 0$, which are distinct real numbers. The corresponding eigenvectors are

$$v = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Hence, the general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 e^{-4t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The constants are given from the linear system

$$\left[\begin{array}{cc|c} 3 & 1 & 5 \\ -1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right].$$

Hence, the solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-4t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Explicit solutions in the case of distinct real eigenvalues: To solve the system

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

do the following

1. Find the eigenvalues λ, μ of A.
2. If $\lambda \neq \mu$ are real numbers, then find the corresponding eigenvectors v, w.
3. Find the constants C_1, C_2 such that
$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = C_1 v + C_2 w.$$
4. The solution is
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda t} v + C_2 e^{\mu t} w.$$

If the eigenvalues λ, μ are not real numbers or if $\lambda = \mu$, then the procedure is similar but a bit more tricky. We will not consider these cases here.

Example 3

Find the solution of the system

$$\frac{d}{dt} x_1(t) = x_1(t) + 4x_2(t) \qquad x_{10} = 4$$

$$\frac{d}{dt} x_2(t) = x_1(t) + x_2(t) \qquad x_{20} = 8.$$

First, we observe that the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ are $\lambda = 3, \mu = -1$, which are distinct real numbers. The corresponding eigenvectors are

$$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad w = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Hence, the general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

The constants are given from the linear system

$$\left[\begin{array}{cc|c} 2 & 2 & 4 \\ 1 & -1 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \end{array}\right].$$

Hence the solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{3t} \begin{bmatrix} 10 \\ 5 \end{bmatrix} - e^{-t} \begin{bmatrix} 6 \\ -3 \end{bmatrix}.$$

Observation III: Stability of zero

Consider the system of differential equations

$$\frac{d}{dt}x(t) = Ax(t) \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

and assume that λ is an eigenvalue of A with corresponding eigenvector v. Then we know that

$$x(t) = e^{\lambda t} v$$

is a solution. If λ is a real number and $\lambda > 0$, then this solution grows in time, but if $\lambda < 0$ then the solution decays to zero. If $\lambda = a + b\mathrm{i}$ is a complex number, then we use Euler's formula

$$e^{\lambda t} = e^{(a+b\mathrm{i})t} = e^{at}[\cos(bt) + \mathrm{i}\sin(bt)].$$

We see that the solution grows if the real part Re(λ)=$a > 0$, and the solution decays to zero if the real part Re(λ)=$a < 0$.

Fact: If all the eigenvalues of the matrix A have negative real part, then all the solutions to the system of differential equations

$$\frac{d}{dt}x(t) = Ax(t), \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

decay to zero as $t \to \infty$. If one eigenvalue has positive real part, then there is at least one solutions that does not decay to zero. (In fact, there is a solution that grows to infinity in an appropriate norm).
Note that we make no statement about the case of eigenvalues with zero real part.

Examples revisited

1. In the first example above, the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$$

has the eigenvalues $\lambda = 3 > 0$ and $\mu = -3 < 0$. Hence there is one solution that grows to infinity, namely

$$x(t) = e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The other decays to zero, since

$$\lim_{t\to\infty} e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

However, then general solution grows, since

$$\lim_{t\to\infty} \left\{ C_1 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \begin{bmatrix} \infty \\ \infty \end{bmatrix}.$$

2. In Newton's law of cooling, we have the matrix

$$A = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$$

with eigenvalues $\lambda = -4, < 0$ and $\mu = 0$. In this case, the box above does not apply. We see that we have the constant solution

$$x(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

that neither decays to zero nor grows to infinity.

3. In the third example above, we had one positive and one negative eigenvalue, hence there is one solution that grows to infinity.

Example 4

Consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 8 & -5 \end{bmatrix}.$$

The eigenvalues are $\lambda = -1, \mu = -3$. Both are real and negative, hence all solutions of

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 8 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

converge to zero.

Example 5

Consider the matrix

$$A = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix}.$$

The eigenvalues are $\lambda = -1+4\mathrm{i}, \mu = -1-4\mathrm{i}$. Both have negative real part, hence all solutions of

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

converge to zero.

Practice problems

I) Find the explicit solution of the linear system of differential equations together with initial conditions.

1.
$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

2.
$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

3.
$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

II) In the following cases, determine whether all solutions of the system $\frac{d}{dt}x = Ax$ converge to zero or not. [Hint: find the real parts of the eigenvalues of A.]

(a) $\begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 4 \\ -3 & 7 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$ (f) $\begin{bmatrix} -2 & 5/2 \\ -1/2 & 0 \end{bmatrix}$.

Solutions to practice problems

I) Explicit solutions

1. The eigenvalues of the matrix are $\lambda = 3$ and $\mu = -1$. The corresponding eigenvectors are
$$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad w = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$
The initial condition gives the linear system
$$C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix},$$
which has the solution $C_1 = 3, C_2 = -2$. Hence, the solution is
$$\begin{bmatrix} x \\ y \end{bmatrix} = 3e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2e^{-t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

2. The eigenvalues of the matrix are $\lambda = 3$ and $\mu = 2$. The corresponding eigenvectors are
$$v = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
The initial condition gives the linear system
$$C_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
which has the solution $C_1 = 1, C_2 = -3$. Hence, the solution is
$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{3t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} - 3e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

3. The eigenvalues of the matrix are $\lambda = -6$ and $\mu = -2$. The corresponding eigenvectors are
$$v = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
The initial condition gives the linear system
$$C_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$
which has the solution $C_1 = -1, C_2 = 2$. Hence, the solution is
$$\begin{bmatrix} x \\ y \end{bmatrix} = -e^{-6t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + 2e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

II) Stability

(a) The eigenvalues are $\lambda = 3 + 2\mathrm{i}$ and $\mu = 3 - 2\mathrm{i}$. Both have positive real part, hence no solution except for zero converges to zero.

(b) The eigenvalues are $\lambda = 3, \mu = -3$. One is real and positive, hence there is a solution that does not converge to zero.

(c) The eigenvalues are $\lambda = 5, \mu = 1$. Both are real and positive, hence all solutions (except zero) grow to infinity.

(d) The eigenvalues are $\lambda = 1+2\mathrm{i}, \mu = 1-2\mathrm{i}$. Both have positive real part, hence all solutions (except zero) grow to infinity.

(e) The eigenvalues are $\lambda = \sqrt{34}, \mu = -\sqrt{34}$. One is real and positive, hence there is a solution that does not converge to zero.

(e) The eigenvalues are $\lambda = -1 + \mathrm{i}/2, \mu = -1 - \mathrm{i}/2$. Both have negative real part, hence all solutions converge to zero.

Systems of differential equations II: the nonlinear case

There are some similarities and many differences between single differential equations and systems of differential equations and how they are treated. Here, we first give a general recipe for how to study systems of two equations, and then we summarize similarities and differences in a table.

A system of two differential equations can be written in the form

$$\begin{aligned}\frac{d}{dt}x &= F(x,y)\\ \frac{d}{dt}y &= G(x,y),\end{aligned}$$

where $x = x(t), y = y(t)$ are the two functions that we are looking for. In general, there is no explicit solution available, i.e., the functions $x(t)$ and $y(t)$ cannot be written down. Nonetheless, we can find out the general shape of solutions. This is done in two parts:

1. Phase plane (as explained in the textbook in Sections 5.6-5.8)
2. Linear stability analysis (not in the textbook, but mostly in the previous section of the lecture notes).

Definitions

The x-**nullcline** is the set of all points (x, y) where $x(t)$ does not change, i.e., $dx/dt = 0$, or $F(x, y) = 0$.
The y-**nullcline** is the set of all points (x, y) where $y(t)$ does not change, i.e., $dy/dt = 0$, or $G(x, y) = 0$.
A **steady state** or **equilibrium** is a point where neither x nor y change, i.e., $dx/dt = 0$ AND $dy/dt = 0$ or, equivalently, $F(x, y) = 0$ AND $G(x, y) = 0$.
A **direction arrow** is a vector that indicates in which direction the solution will go from a given point. The direction arrow at the point (x, y) has the coordinates $(F(x, y), G(x, y))$.
The **linearization** of the system at a steady state (x^*, y^*) is given by the linear system

$$\frac{d}{dt}\begin{bmatrix} X \\ Y \end{bmatrix} = J(x^*,y^*)\begin{bmatrix} X \\ Y \end{bmatrix}, \qquad J(x,y) = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix},$$

where $X(t), Y(t)$ are the small deviations from x^*, y^*, i.e., $x(t) = x^* + X(t), y(t) = y^* + Y(T)$.

Recipe

Given the system

$$\begin{aligned} \frac{d}{dt}x &= F(x,y), \\ \frac{d}{dt}y &= G(x,y), \end{aligned}$$

do the following steps.

1. Phase-Plane

   (a) Find the nullcline of x, i.e., the set of points where $dx/dt = 0$, or equivalently $F(x,y) = 0$.

   (b) Find the nullcline of y, i.e., the set of points where $dy/dt = 0$, or equivalently $G(x,y) = 0$.

   (c) Draw these two sets in the x-y plane.

   (d) Find the steady states, i.e., the points where $F(x,y) = 0$ AND $G(x,y) = 0$, i.e., the intersection of the nullclines.

   (e) On each of the nullclines, draw the direction arrows.

   (f) In each of the regions in space in between the nullclines, draw the direction arrows.

2. Linear stability analysis

   (a) Find the Jacobi matrix

   $$J(x,y) = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}.$$

   (b) Evaluate the Jacobi matrix at each stationary point and find the real parts of the eigenvalues.

   (c) If the real parts are negative, then the stationary point is stable; if one real part is positive, it is unstable.

3. For a given initial condition, draw a solution into the phase plane. Then plot the two components of the solution as a function of time.

Table 1: Summary and comparison of one- and two-dimensional differential equations and the corresponding solution techniques.

	1-D	2-D
Linear equation	$\frac{dx}{dt} = rx$	$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ $A = \begin{bmatrix} a & b \\ c & e \end{bmatrix}$
Solution to linear equation	$x(t) = e^{rt} x_0$	$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda t} v + C_2 e^{\mu t} w$ $Av = \lambda v,\ Aw = \mu w$
Nonlinear equation	$\frac{dx}{dt} = f(x)$	$\frac{dx}{dt} = F(x,y)$ $\frac{dy}{dt} = G(x,y)$
Explicit solution	Separation of variables $\int \frac{dx}{f(x)} = \int dt$	Generally not available
Graphical Solution	Phase-line diagram, Slope field	Phase-plane
Steady state	$f(x^*) = 0$	$F(x^*, y^*) = 0, G(x^*, y^*) = 0$
Stability	$f'(x^*) < 0$	Eigenvalues of $J(x^*, y^*)$ have negative real part $J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}$

1. Find the volume of the solid obtained by rotating the area bounded by $y = 4x - x^2$, $y = 3$, $x = 1$ and $x = 3$ about the x-axis.

   Sol. See Figure 13 .

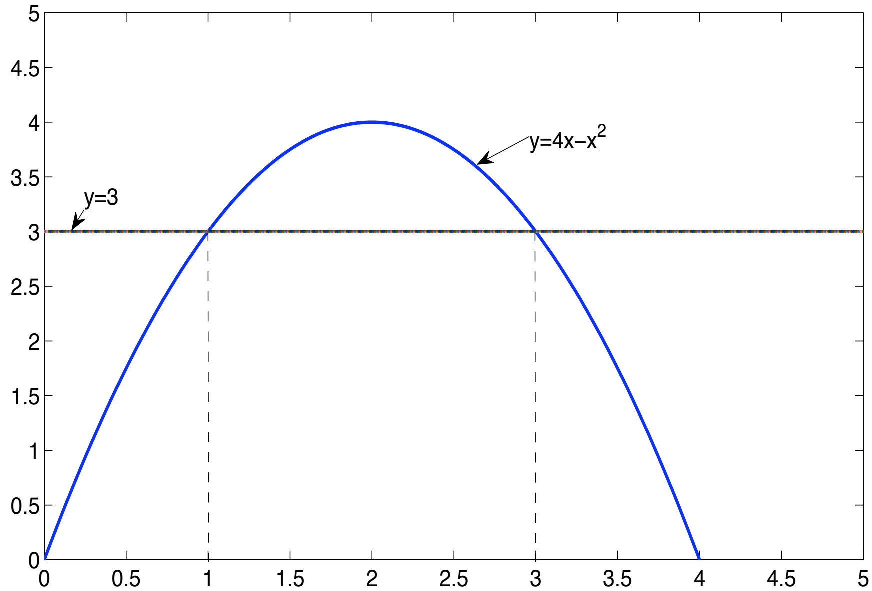

Figure 13: The illustrative graph of Question 1.

Denote $A_1(x) = 4x - x^2$ and $A_2(x) = 3$. The volume of the solid is given by

$$\begin{aligned} V &= V_1 - V_2 \\ &= \int_1^3 \pi(A_1(x))^2 dx - \int_1^3 \pi(A_2(x))^2 dx \\ &= \int_1^3 \pi(4x - x^2)^2 dx - \int_1^3 \pi \cdot 3^2 dx \\ &= \int_1^3 \pi(16x^2 - 8x^3 + x^4) dx - \int_1^3 9\pi dx \\ &= \pi(\frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5)\Big|_1^3 - 9\pi x\Big|_1^3 \\ &= \frac{136}{15}\pi. \end{aligned}$$

2. Consider the system

$$\begin{aligned} x' &= \sqrt{3}x + 2y \\ y' &= 11x - \sqrt{3}y \end{aligned}$$

(a) Show that $(0,0)$ is the only equilibrium.
(b) Write down the Jacobian matrix.
(c) Show that the eigenvalues are $\lambda = \pm 5$.
(d) For each eigenvalue, find the corresponding eigenvectors.

Sol.

(a) Set $x' = 0$ and $y' = 0$. Then we have

$$\begin{aligned} \sqrt{3}x + 2y &= 0 \\ 11x - \sqrt{3}y &= 0 \end{aligned}$$

Solving the above linear system for x and y, we obtain the unique solution $x = 0$ and $y = 0$.

(b) The Jacobian matrix is given by

$$J = \begin{bmatrix} \sqrt{3} & 2 \\ 11 & -\sqrt{3} \end{bmatrix}.$$

(c) We have

$$\begin{aligned} \det(J - \lambda I) = J &= \begin{bmatrix} \sqrt{3} - \lambda & 2 \\ 11 & -\sqrt{3} - \lambda \end{bmatrix}. \\ &= \lambda^2 - 25. \end{aligned}$$

Thus $\lambda = \pm 5$.

(d) For $\lambda_1 = 5$, we have

$$\left[\begin{array}{cc|c} \sqrt{3} - 5 & 2 & 0 \\ 11 & -\sqrt{3} - 5 & 0 \end{array}\right] \xrightarrow{(\sqrt{3}+5)R_1} \left[\begin{array}{cc|c} -22 & 2(\sqrt{3}+5) & 0 \\ 11 & -\sqrt{3} - 5 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1}$$

$$\left[\begin{array}{cc|c} -11 & \sqrt{3}+5 & 0 \\ 11 & -\sqrt{3} - 5 & 0 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} -11 & \sqrt{3}+5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Set $x_2 = t$. From the first row, we have $-11x_1 + (\sqrt{3}+5)x_2 = 0$. Solving for x_1 gives $x_1 = \frac{\sqrt{3}+5}{11}t$. Thus, the corresponding eigenvectors are

$$v_1 = \begin{bmatrix} \frac{\sqrt{3}+5}{11} \\ 1 \end{bmatrix} t, \quad (t \neq 0).$$

For $\lambda_2 = -5$, we have

$$\left[\begin{array}{cc|c} \sqrt{3} + 5 & 2 & 0 \\ 11 & -\sqrt{3} + 5 & 0 \end{array}\right] \xrightarrow{(\sqrt{3}-5)R_1} \left[\begin{array}{cc|c} -22 & 2(\sqrt{3}-5) & 0 \\ 11 & -\sqrt{3} + 5 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1}$$

$$\left[\begin{array}{cc|c} -11 & \sqrt{3}-5 & 0 \\ 11 & -\sqrt{3} + 5 & 0 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} -11 & \sqrt{3}-5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Set $x_2 = t$. From the first row, we have $-11x_1 + (\sqrt{3}-5)x_2 = 0$. Solving for x_1 gives $x_1 = \frac{\sqrt{3}-5}{11}t$. Thus, the corresponding eigenvectors are

$$v_2 = \begin{bmatrix} \frac{\sqrt{3}-5}{11} \\ 1 \end{bmatrix} t, \quad (t \neq 0).$$

3. Evaluate $\int_1^4 e^{\sqrt{x}}dx$.

Sol. Let $u = \sqrt{x}$, then $du = \frac{1}{2}\frac{1}{\sqrt{x}}dx$, i.e. $dx = 2udu$. The integration limits are changed from $x = 1$ and $x = 4$ to $u = 1$ and $u = 2$. So using integration by parts, we have

$$\begin{aligned} \int_1^4 e^{\sqrt{x}}dx &= \int_1^2 e^u \cdot 2udu = 2\int_1^2 ude^u = 2ue^u\Big|_1^2 - 2\int_1^2 e^u du \\ &= 2ue^u\Big|_1^2 - 2e^u\Big|_1^2 = 2e^2. \end{aligned}$$

4. Evaluate $\int_1^2 \ln(x^2e^x)dx$.

Sol.

$$\int_1^2 \ln(x^2e^x)dx = \int_1^2 (\ln x^2 + \ln e^x)dx = \int_1^2 (2\ln x + x)dx = 2\int_1^2 \ln x dx + \int_1^2 xdx.$$

Using integration by parts for the first integral with $u = \ln x$, $v' = 1$, we have

$$\begin{aligned} \int_1^2 \ln(x^2e^x)dx &= 2x\ln x\Big|_1^2 - 2\int_1^2 x \cdot \frac{1}{x}dx + \int_1^2 xdx = 2x\ln x\Big|_1^2 - 2\int_1^2 dx + \int_1^2 xdx \\ &= 2x\ln x\Big|_1^2 - 2x\Big|_1^2 + \frac{1}{2}x^2\Big|_1^2 = 4\ln 2 - \frac{1}{2}. \end{aligned}$$

5. Evaluate $\int \frac{2x-1}{(x+4)(x+1)}dx$.

Sol.

$$\frac{2x-1}{(x+4)(x+1)} = \frac{A}{x+4} + \frac{B}{x+1} = \frac{A(x+1)+B(x+4)}{(x+4)(x+1)} = \frac{(A+B)x+(A+4B)}{(x+4)(x+1)}$$

Setting $A + B = 2$ and $A + 4B = -1$, we have $A = 3$ and $B = -1$. Hence

$$\int \frac{2x-1}{(x+4)(x+1)}dx = \int \frac{3}{x+4} + \frac{-1}{x+1}dx = 3\ln|x+4| - \ln|x+1| + C.$$

6. Evaluate $\int \frac{x^2+1}{x^2+3x+2}dx$.

Sol. Note that $x^2 + 3x + 2 = (x+1)(x+2)$. Using long division, we have

$$\frac{x^2+1}{x^2+3x+2} = 1 - \frac{3x+1}{x^2+3x+2}.$$

Thus

$$\int \frac{x^2+1}{x^2+3x+2}dx = \int \frac{(x^2+3x+2)-3x-1}{x^2+3x+2} = \int dx - \int \frac{3x+1}{x^2+3x+2}dx$$

However,

$$\frac{3x+1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)} = \frac{(A+B)x+(2A+B)}{(x+1)(x+2)}$$

Setting $A+B=3$ and $2A+B=1$, we have $A=-2$, and $B=5$. Finally

$$\begin{aligned}\int \frac{x^2+1}{x^2+3x+2}dx &= \int dx - \int \frac{3x+1}{x^2+3x+2}dx \\ &= \int dx - \int \frac{-2}{x+1} + \frac{5}{x+2}dx \\ &= x + 2\ln|x+1| - 5\ln|x+2| + C.\end{aligned}$$

7. Find the area enclosed by the following two functions $f(x)=(x-1)^2-1$, $g(x)=-x+2$.
Sol. The bounding curves are graphed in Figure 14.

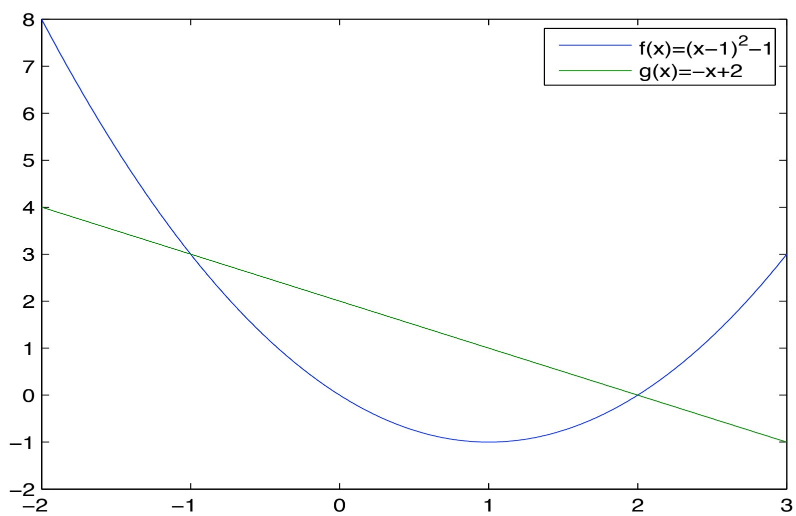

Figure 14: The graphs of $f(x)=(x-1)^2-1$ and $g(x)=-x+2$ in Question 7.

To find the points where the two curves intersect, we solve

$$(x-1)^2-1=-x+2 \Longleftrightarrow (x+1)(x-2)=0$$

Therefore, $x=-1$ and $x=2$ are the x-coordinates of the points of intersection. The area of enclosed by $f(x)$ and $g(x)$ is given by

$$\begin{aligned}\int_{-1}^{2}|f(x)-g(x)|dx &= \int_{-1}^{2}[g(x)-f(x)]dx\\ &= \int_{-1}^{2}[-x+2-(x-1)^2+1]dx\\ &= \int_{-1}^{2}[-x+2-x^2+2x-1+1]dx\\ &= \int_{-1}^{2}[-x^2+x+2]dx = [-\frac{1}{3}x^3+\frac{1}{2}x^2+2x]\Big|_{-1}^{2} = \frac{9}{2}.\end{aligned}$$

8. Does the integral $\int_0^2 \frac{1}{(1-x)^{1/3}}dx$ converge? If so, to which value?

   Sol. The function is not defined at $x=1$, so we need to split the integral.

$$\begin{aligned}\int_0^2 \frac{1}{(1-x)^{1/3}}dx &= \int_0^1 \frac{1}{(1-x)^{1/3}}dx + \int_1^2 \frac{1}{(1-x)^{1/3}}dx\\ &= -\frac{3}{2}(1-x)^{\frac{2}{3}}\Big|_0^1 - \frac{3}{2}(1-x)^{\frac{2}{3}}\Big|_1^2\\ &= -0+\frac{3}{2}-\frac{3}{2}+0=0.\end{aligned}$$

9. Does the integral $\int_0^2 \frac{1}{(x-1)^4}dx$ converge? If so, to which value?

   Sol. The function is not defined at $x=1$, so we need to split the integral.

$$\begin{aligned}\int_0^2 \frac{1}{(x-1)^4}dx &= \int_0^1 \frac{1}{(x-1)^4}dx + \int_1^2 \frac{1}{(x-1)^4}dx\\ &= -\frac{1}{3}(x-1)^{-3}\Big|_0^1 - \frac{1}{3}(x-1)^{-3}\Big|_1^2\\ &= -\infty-\frac{1}{3}-\frac{1}{3}+\infty.\end{aligned}$$

   So this integral doesn't converge. (Note that we wouldn't have seen this if we'd tried to integrate without splitting.)

10. Does the integral $\int_0^2 \frac{1}{(x-1)^{2/5}}dx$ converge? If so, to which value?

    Sol. The function is not defined at $x=1$, so we need to split the integral.

$$\begin{aligned}\int_0^2 \frac{1}{(x-1)^{2/5}}dx &= \int_0^1 \frac{1}{(x-1)^{2/5}}dx + \int_1^2 \frac{1}{(x-1)^{2/5}}dx\\ &= \frac{5}{3}(x-1)^{\frac{3}{5}}\Big|_0^1 + \frac{5}{3}(x-1)^{\frac{3}{5}}\Big|_1^2\\ &= 0-\frac{5}{3}+\frac{5}{3}-0=0.\end{aligned}$$

11. Does the integral $\int_0^\infty \frac{1}{\sqrt{x+1}}dx$ converge? If so, to which value?

    Sol.

$$\int_0^\infty \frac{1}{\sqrt{x+1}}dx = 2(x+1)^{\frac{1}{2}}\Big|_0^\infty = \infty - 2.$$

    So this integral diverges.

12. Solve $\frac{dx}{dt} = \sqrt{3t+1}$ with $x(0) = 1$.

    Sol. This differential equation is a pure-time differential equation.

$$\begin{aligned} \frac{dx}{dt} &= \sqrt{3t+1} \\ dx &= (3t+1)^{1/2}dt \\ \int dx &= \int (3t+1)^{1/2}dt. \end{aligned}$$

    Let $u = 3t+1$. Then $du = 3dt$, so we have

$$\begin{aligned} x &= \frac{1}{3}\cdot\frac{2}{3}(3t+1)^{\frac{3}{2}} + C \\ x(t) &= \frac{2}{9}(3t+1)^{\frac{3}{2}} + C. \end{aligned}$$

    Applying the initial condition $x(0) = 1$, we obtain $C = \frac{7}{9}$. Thus the solution is

$$x(t) = \frac{2}{9}(3t+1)^{\frac{3}{2}} + \frac{7}{9}.$$

13. Solve $\frac{dx}{dt} = 5 - 16t^2$ with $x(3) = -11$.

    Sol. This differential equation is a pure time differential equation.

$$\begin{aligned} \frac{dx}{dt} &= 5 - 16t^2 \\ dx &= (5 - 16t^2)dt \\ \int dx &= \int (5 - 16t^2)dt \\ x &= 5t - \frac{16}{3}t^3 + C. \end{aligned}$$

    Applying the initial condition $x(3) = -11$, we obtain $C = 118$. Thus, the solution is

$$x(t) = 5t - \frac{16}{3}t^3 + 118.$$

14. Solve $\frac{dy}{dx} = 3x^2e^{2y}$ with $y(0) = 0$.

Sol. This non-autonomous differential equation is separable.

$$\begin{aligned} \frac{dy}{dx} &= 3x^2e^{2y} \\ e^{-2y}dy &= 3x^2dx \\ \int e^{-2y}dy &= \int 3x^2dx \\ -\frac{1}{2}e^{-2y} &= x^3 + C \\ e^{-2y} &= -2x^3 - 2C \\ -2y &= \ln(-2x^3 - 2C) \\ y &= -\frac{1}{2}\ln(-2x^3 - 2C). \end{aligned}$$

Applying the initial condition $y(0) = 0$, we obtain $C = -\frac{1}{2}$. So the solution is

$$y(x) = -\frac{1}{2}\ln(1 - 2x^3).$$

15. Solve $\frac{dy}{dx} = \frac{2x}{y+e^{5y}}$ with $y(2) = 0$.

Sol. This non-autonomous differential equation is separable.

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x}{y + e^{5y}} \\ (y + e^{5y})dy &= 2xdx \\ \int (y + e^{5y})dy &= \int 2xdx \\ \frac{y^2}{2} + \frac{e^{5y}}{5} &= x^2 + C. \end{aligned}$$

Note that we can't solve for y explicitly. Applying the initial condition $y(2) = 0$, we get $C = -\frac{19}{5}$. So the solution is implicitly given by

$$\frac{y^2}{2} + \frac{e^{5y}}{5} = x^2 - \frac{19}{5}.$$

16. Suppose that

$$\frac{dy}{dx} = y(1 - y)(y - 2)$$

(a) Find the equilibria of this differential equation.

(b) Graph $\frac{dy}{dx}$ as a function of y and use your graph to discuss the stability of the equilibria.

(c) Draw the phase-line diagram.

(d) Use the derivative test to discuss the stability of equilibria.

(e) Sketch the solution with the initial conditions $y(0) = 0.5$ and $y(0) = 1.5$ respectively.

Sol. This differential equation is autonomous.

(a) The equilibria of this differential equation are given by solving

$$y(1-y)(y-2) = 0 \Longleftrightarrow y_1 = 0,\ y_2 = 1,\ y_3 = 2.$$

(b) The function $\frac{dy}{dx}$ as a function of y is graphed in Figure 15. From the graph of $\frac{dy}{dx}$, the slope at $y_1 = 0$ is negative, so $y_1 = 0$ should be stable; the slope at $y_2 = 1$ is positive, so $y_2 = 1$ should be unstable; the slope at $y_3 = 2$ is negative, so $y_3 = 2$ should be stable.

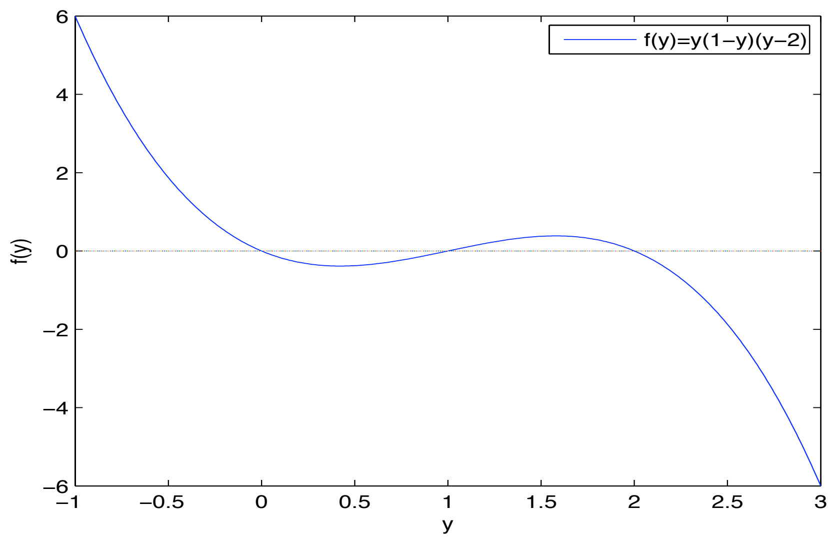

Figure 15: The graph of $\frac{dy}{dx}$ as a function of y in part (b) of Question 16.

(c) The phase-line diagram is given in Figure 16.

(d) Let $f(y) = y(1-y)(y-2)$, i.e., $f(y) = -y^3 + 3y^2 - 2y$. Then $f'(y) = -3y^2 + 6y - 2$ and

$$\begin{aligned} f'(y_1) &= f'(0) = -2 < 0, \text{ so } y_1 = 0 \text{ is stable;} \\ f'(y_2) &= f'(1) = 1 > 0, \text{ so } y_2 = 1 \text{ is unstable;} \\ f'(y_3) &= f'(2) = -2 < 0, \text{ so } y_3 = 2 \text{ is stable.} \end{aligned}$$

(e) The solution with initial conditions $y(0) = 0.5$ and $y(0) = 1.5$ are sketched in Figure 17.

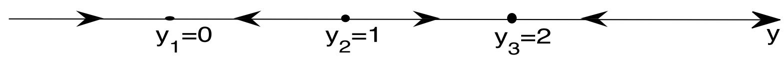

Figure 16: The phase-line diagram in part (c) of Question 16.

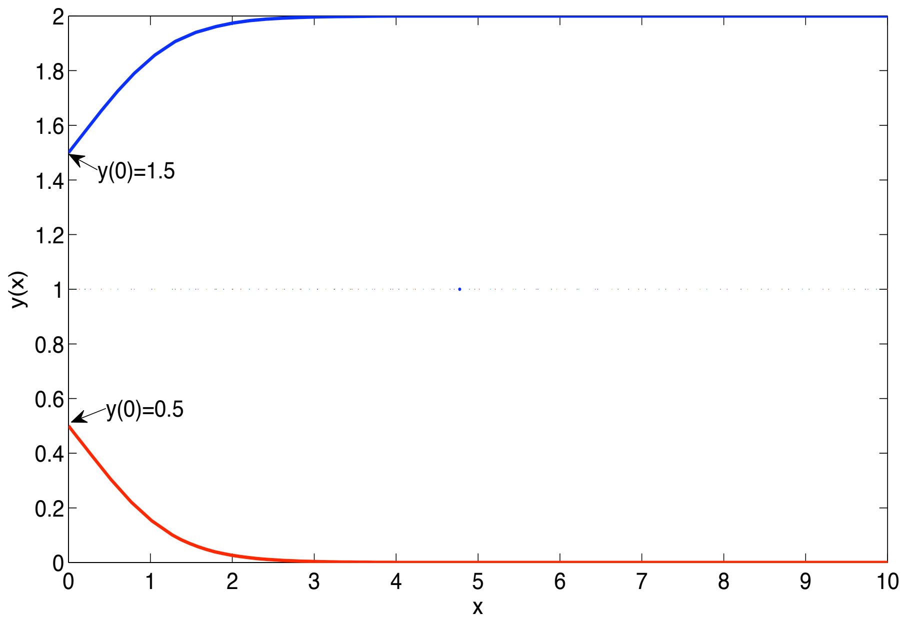

Figure 17: The solution curves with initial conditions $y(0) = 0.5$ and $y(0) = 1.5$ in part (e) of Question 16.

17. Considering the following predator-prey system:

$$\begin{cases} \frac{dN}{dt} = aN(1 - \frac{N}{K}) - bNP \\ \frac{dP}{dt} = cNP - dP \end{cases}$$

where $N = N(t)$ is the prey density at time t and $P = P(t)$ is the predator density at time t. The constants a, b, c, d and K are positive. Furthermore, assume that $d/c < K$.

(a) Find all of equilibrium points.

(b) Find the Jacobian matrix of the system in general.

(c) Evaluate the Jacobian matrix at the nontrivial equilibrium (N^*, P^*) with $N^* > 0$ and $P^* > 0$. Find the eigenvalues and determine whether this point is stable or not.

Sol.

(a) The equilibrium points will be given by solving

$$\begin{aligned} aN(1-\frac{N}{K}) - bNP &= 0 \Longleftrightarrow N[a(1-\frac{N}{K}) - bP] = 0 \Longleftrightarrow N = 0 \text{ or } a(1-\frac{N}{K}) - bP = 0 \\ cNP - dP &= 0 \Longleftrightarrow (cN-d)p = 0 \Longleftrightarrow N = \frac{d}{c} \text{ or } P = 0 \end{aligned}$$

So the equilibrium points are

$$\begin{aligned} (N_1, P_1) &= (0,0) \\ (N_2, P_2) &= \Big(\frac{d}{c}, \frac{a}{b}(1-\frac{1}{K}\frac{d}{c})\Big), \\ (N_3, P_3) &= (K, 0). \end{aligned}$$

(b) Let $f(N,P) = aN(1-\frac{N}{K}) - bNP = aN - \frac{a}{K}N^2 - bNP$ and $g(N,P) = cNP - dP$. Thus the Jacobian matrix of the system in general is given by

$$J(N,P) = \begin{bmatrix} a - 2\frac{a}{K}N - bP & -bN \\ cP & cN - d \end{bmatrix}.$$

(c) Since $d/c < K$ and $P_2 = \frac{a}{b}(1-\frac{1}{K}\frac{d}{c}) > 0$, (N_2, P_2) is the only nontrivial equilibrium. Evaluating the Jacobian matrix $J(N,P)$ at (N_2, P_2), we have

$$\begin{aligned} J(N_2, P_2) &= \begin{bmatrix} a - 2\frac{a}{K}N_2 - bP_2 & -bN_2 \\ cP_2 & cN_2 - d \end{bmatrix} = \begin{bmatrix} a - 2\frac{a}{K}\frac{d}{c} - b\frac{a}{b}(1-\frac{1}{K}\frac{d}{c}) & -b\frac{d}{c} \\ c\frac{a}{b}(1-\frac{1}{K}\frac{d}{c}) & c\frac{d}{c} - d \end{bmatrix} \\ &= \begin{bmatrix} -\frac{ad}{Kc} & -\frac{bd}{c} \\ \frac{ac}{b} - \frac{ad}{bK} & 0 \end{bmatrix} \\ \det(J = \lambda I) &= \begin{bmatrix} -\frac{ad}{Kc} - \lambda & -\frac{bd}{c} \\ \frac{ac}{b} - \frac{ad}{bK} & -\lambda \end{bmatrix} \\ &= \lambda^2 + \frac{ad}{Kc}\lambda + \frac{bd}{c}\left(\frac{ac}{b} - \frac{ad}{bK}\right). \end{aligned}$$

Then the eigenvalues are given by

$$\lambda = -\frac{1}{2}\frac{ad}{Kc} \pm \sqrt{\frac{1}{4}\frac{(ad)^2}{(Kc)^2} - ad\left(1 - \frac{d}{c}\frac{1}{K}\right)}.$$

Note that $d/c < K$ means that $1-\frac{d}{c}\frac{1}{K} > 0$, and then $\frac{1}{4}\frac{(ad)^2}{(Kc)^2}-ad(1-\frac{d}{c}\frac{1}{K}) < \frac{1}{4}\frac{(ad)^2}{(Kc)^2}$, which implies that λ will always have negative real parts. Hence, this nontrivial equilibrium (N_2, P_2) is stable.

18. Find the tangent plane to the surface

$$z = f(x,y) = 4x^2 + y^2$$

at the point $(1, 2, 8)$.

Sol. The partial derivatives are given by

$$\begin{aligned} \frac{\partial f}{\partial x} &= 8x \\ \frac{\partial f}{\partial y} &= 2y. \end{aligned}$$

The function satisfies $z_0 = f(1,2) = 4\cdot 1^2 + 2^2 = 8$, and $\frac{\partial f}{\partial x}(1,2) = 8$, $\frac{\partial f}{\partial y}(1,2) = 4$. Thus, the tangent plane is given by

$$\begin{aligned} z - z_0 &= \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) \\ z - 8 &= 8(x-1) + 4(y-2) \\ z &= 8x + 4y - 8. \end{aligned}$$

19. Let

$$A = \begin{bmatrix} -4 & -2 \\ 1 & -1 \end{bmatrix}$$

(a) Show that the eigenvalues of A are -3 and -2.
(b) Find the solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -4x - 2y \\ \frac{dy}{dt} &= x - y \end{aligned}$$

if $x(0) = 1$ and $y(0) = 2$.
(c) Draw the phase portrait of the system of differential equations given in (b).

Sol.

(a) We have

$$\begin{aligned} \det(J - \lambda I) &= \begin{bmatrix} -4-\lambda & -2 \\ 1 & -1-\lambda \end{bmatrix} \\ &= \lambda^2 + 5\lambda + 6 \\ &= (\lambda+2)(\lambda+3) \end{aligned}$$

Hence, $\lambda_1 = -3$ and $\lambda_2 = -2$.

(b) For $\lambda_1 = -3$, we have

$$\left[\begin{array}{cc|c} -4+3 & -2 & 0 \\ 1 & -1+3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & -2 & 0 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Set $x_2 = t$. From the first row, we have $-x_1 - 2x_2 = 0$. Then $x_1 = -2t$. So a corresponding eigenvector is given by

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

For $\lambda_1 = -2$, we have

$$\left[\begin{array}{cc|c} -4+2 & -2 & 0 \\ 1 & -1+2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -2 & -2 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2} * R_1 + R_2} \left[\begin{array}{cc|c} -2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Set $x_2 = t$. From the first row, we have $-2x_1 - 2x_2 = 0$. Then $x_1 = -t$. So a corresponding eigenvector is given by

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The general solution is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{-3t} v_1 + C_2 e^{-2t} v_2 = C_1 e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Applying initial conditions, we have

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2C_1 - C_2 \\ C_1 + C_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Solving this gives us that $C_1 = -3$ and $C_2 = 5$. The solution is finally given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 5e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6e^{-3t} - 5e^{-2t} \\ -3e^{-3t} + 5e^{-2t} \end{bmatrix}.$$

(c) The phase portrait of the system of differential equations given in (b) is graphed in Figure 18.

20. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -5 \\ 2 & 5 & -6 \end{bmatrix}$. Solve the system of linear equations $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

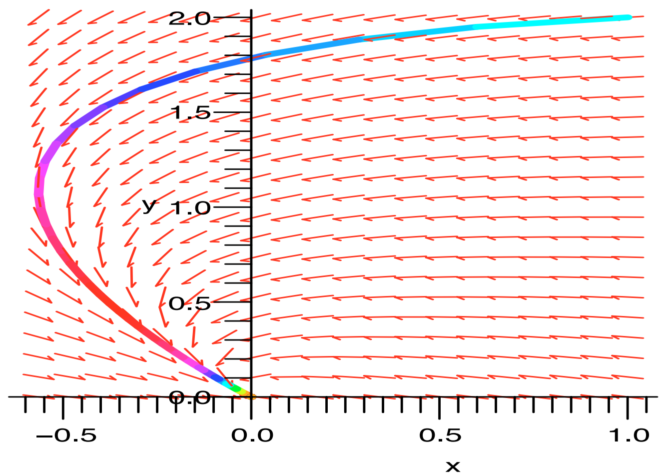

Figure 18: The phase portrait in part (c) of Question 19.

Sol. Note that $\det(A) = 1\cdot 3\cdot(-6) + 1\cdot(-5)\cdot 2 - (-1)\cdot 3\cdot 2 - 1\cdot(-5)\cdot 5 = 3 \neq 0$, so A is invertible.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -5 & 0 & 1 & 0 \\ 2 & 5 & -6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-2)*R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -5 & 0 & 1 & 0 \\ 0 & 3 & -4 & -2 & 0 & 1 \end{array}\right] \xrightarrow{(-1)*R_2+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -5 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 3 & -5 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow{5*R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 3 & 0 & -10 & -4 & 5 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}*R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{4}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow{(-1)*R_2+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{4}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} \frac{7}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{10}{3} & -\frac{4}{3} & \frac{5}{3} \\ -2 & -1 & 1 \end{bmatrix}.$$

The solution is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{10}{3} & -\frac{4}{3} & \frac{5}{3} \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ -\frac{5}{3} \\ -1 \end{bmatrix}.$$

Additional Problems

Problems here in this section are from the book titled "Calculus for Biology and Medicine" by Claudia Neuhauser.

Find the volumes of solids obtained by rotating the region bounded by the given curves about the x-axis.

1. $$y = 4 - x^2, \;\; y = 0, \;\; x = 0 \;\; \text{(in the first quadrant)}$$

2. $$y = \sqrt{\sin x}, \;\; 0 \le x \le \pi, \;\; y = 0.$$

3. $$y = 1 - x^2, \;\; y = 1 + x^2, \;\; 0 \le x \le 1$$

4. $$y = \frac{1}{x}, \;\; x = 0, \;\; y = 1, \;\; y = 2, \;\; \text{(in the first quadrant)}$$

5. $$y = e^x, \;\; y = e^{-x}, \;\; 0 \le x \le 2$$

Evaluate each integral

6. $\int \frac{2x-1}{x^2+5x+4} dx$
7. $\int \frac{1}{x^2-x-6} dx$
8. $\int \frac{x^2+1}{x^2+3x+2} dx$
9. $\int \frac{x^3+1}{x^2+3}$
10. $\int_3^5 \frac{x}{x-1} dx$
11. $\int_2^3 \frac{1}{1-x^2} dx$

Determine whether each integral is convergent. If the integral is convergent, compute its value.

12. $$\int_0^\infty 3e^{-3x} dx$$

13. $$\int_0^\infty \frac{2}{1+x^2} dx$$

14. $$\int_1^\infty \frac{1}{x^{3/2}} dx$$

15.
$$\int_0^9 \frac{1}{\sqrt{9-x}}dx$$

16.
$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}dx$$

17.
$$\int_1^{\infty} \frac{1}{x^3}dx$$

18.
$$\int_0^4 \frac{1}{x^4}dx$$

19.
$$\int_0^2 \frac{1}{(x-1)^{1/3}}dx$$

20.
$$\int_0^{\infty} \frac{1}{\sqrt{x+1}}dx$$

21.
$$\int_e^{\infty} \frac{1}{x\ln x}dx$$

Solve

22.
$$\frac{dy}{dx} = y(y-5) \quad \text{with} \quad y_0 = 1 \quad \text{if} \quad x_0 = 0$$

23.
$$\frac{dy}{dx} = 2y(3-y) \quad \text{with} \quad y_0 = 5 \quad \text{if} \quad x_0 = 1$$

24.
$$\frac{dy}{dx} = \frac{x+1}{y} \quad \text{with} \quad y_0 = 2 \quad \text{if} \quad x_0 = 0$$

25.
$$\frac{dy}{dx} = (y+1)e^{-x} \quad \text{with} \quad y_0 = 2 \quad \text{if} \quad x_0 = 0$$

26.
$$\frac{dy}{dx} = \frac{y+1}{x-1} \quad \text{with} \quad y_0 = 5 \quad \text{if} \quad x_0 = 2$$

Solve each system of linear equations.

27.

$$\begin{aligned} 5x_1 - x_2 + 2x_3 &= 6 \\ x_1 + 2x_2 - x_3 &= -1 \\ 3x_1 + 2x_2 - 2x_3 &= 1 \end{aligned}$$

28.

$$\begin{aligned} -2x_1 + 4x_2 - x_3 &= -1 \\ x_1 + 7x_2 + 2x_3 &= -4 \\ 3x_1 - 2x_2 + 3x_3 &= -3 \end{aligned}$$

29.

$$\begin{aligned} -x_1 - 2x_2 + 3x_3 &= -9 \\ 2x_1 + x_2 - x_3 &= 5 \\ 4x_1 - 3x_2 + 5x_3 &= -9 \end{aligned}$$

30.

$$\begin{aligned} x_1 + x_2 &= 3 \\ -x_2 + x_3 &= -1 \\ x_1 + x_3 &= 2 \end{aligned}$$

31.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 - 3x_2 + x_3 &= 8 \end{aligned}$$

32.

$$\begin{aligned} 2x_1 - x_2 &= 3 \\ x_1 - x_2 &= 4 \\ 3x_1 - x_2 &= 1 \end{aligned}$$

33. Determine c such that

$$\begin{aligned} 2x_1 - 3x_2 &= 5 \\ 4x_1 - 6x_2 &= c \end{aligned}$$

has (a) infinitely many solutions and (b) no solutions. (c) Is it possible to choose a number for c so that the system has exactly one solution? Explain your answer.

Evaluate each function at the given point.

34. $f(x,y) = \frac{2x}{x^2+y^2}$ at $(2,3)$
35. $h(x,t) = \exp\left[-\frac{(x-2)^2}{2t}\right]$ at $(1,5)$

Find the largest possible domain and the corresponding range of each function. Determine the equation of the level sets $f(x,y) = c$, together with the possible values of c.

36. $f(x,y) = x^2 + y^2$
37. $f(x,y) = \ln(y - x^2)$

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the given functions

38. $f(x,y) = x^2 y + xy^2$
39. $f(x,y) = (xy)^{3/2} - (xy)^{2/3}$
40. $f(x,y) = \sin(x+y)$
41. $f(x,y) = \cos^2(x^2 - 2y)$
42. $f(x,y) = e^{\sqrt{x+y}}$
43. $f(x,y) = e^x \sin(xy)$
44. $f(x,y) = \ln(2x+y)$

Find the indicated partial derivatives.

45. $f(x,y) = 3x^2 - y - 2y^2$; $f_x(1,0)$
46. $g(x,y) = e^{x+3y}$; $g_y(2,1)$
47. $f(x,z) = \ln(xz)$; $f_z(e,1)$
48. $f(x,y) = \frac{xy}{x^2+z}$; $f_x(-1,2)$

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ for the given functions

49. $f(x,y,z) = x^2 z + yz^2 - xy$
50. $f(x,y,z) = x^3 y^2 z + \frac{x}{yz}$
51. $f(x,y,z) = e^{x+y+z}$
52. $f(x,y,z) = \ln(x+y+z)$

In the following problems, the tangent plane at the indicated point (x_0, y_0, z_0) exist. Find its equations

53. $f(x, y) = 2x^3 + y^2$; $(1, 2)$
54. $f(x, y) = e^{x^2+y^2}$; $(1, 0)$
55. $f(x, y) = \sin(xy)$; $(1, 0)$

Find the linear approximation of $f(x, y)$ at the indicated point (x_0, y_0).

56. $f(x, y) = \sqrt{x} + 2y$; $(1, 0)$
57. $f(x, y) = \tan(x + y)$; $(0, 0)$
58. $f(x, y) = \ln(x^2 + y)$; $(1, 1)$
59. Find the linear approximation of $f(x, y) = e^{x+y}$ at $(0, 0)$ and use it to approximate $f(0.1, 0.05)$. Compare the approximation to $f(0.1, 0.05)$ using a calculator.
60. Find the linear approximation of $f(x, y) = \ln(x^2 - 3y)$ at $(1, 0)$ and use it to approximate $f(1.1, 0.1)$. Compare the approximation to $f(1.1, 0.1)$ using a calculator.

Find the Jacobian matrix for each given function

61. $F(x, y) = \begin{bmatrix} x + y \\ x^2 - y^2 \end{bmatrix}$
62. $F(x, y) = \begin{bmatrix} e^{x-y} \\ e^{x+y} \end{bmatrix}$
63. $F(x, y) = \begin{bmatrix} \cos(x - y) \\ \cos(x + y) \end{bmatrix}$
64. $F(x, y) = \begin{bmatrix} 2x^2y - 3y + x \\ e^x \sin y \end{bmatrix}$
65. $J(x, y) = \begin{bmatrix} 1 & 1 \\ 2x & -2y \end{bmatrix}$
66. $J(x, y) = \begin{bmatrix} e^{x-y} & -e^{x-y} \\ e^{x+y} & e^{x+y} \end{bmatrix}$
67. $J(x, y) = \begin{bmatrix} -\sin(x - y) & \sin(x - y) \\ -\sin(x + y) & -\sin(x + y) \end{bmatrix}$
68. $J(x, y) = \begin{bmatrix} 4xy + 1 & 2x^2 - 3 \\ e^x \sin y & e^x \cos y \end{bmatrix}$

69. Find a linear approximation to

$$F(x,y) = \begin{bmatrix} x^2 - xy \\ 3y^2 - 1 \end{bmatrix}$$

at $(1,2)$. Use your result to find an approximation for $F(1.1,1.9)$ and compare the approximation to the value of $F(1.1,1.9)$ when you use a calculator.