This midterm has ***5 questions*** *on* ***6 pages****, for a total of 30 points.*

Duration: 50 minutes

- Read all the questions carefully before starting to work.
- With the exception of Q1, you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (including all middle names): ______________________________

Student-No: ______________________________

Signature: ______________________________

Question:	1	2	3	4	5	Total
Points:	7	5	5	6	7	30
Score:						

7 marks 1. (a) Write the negation of the following statement:

For every function f if $x = y$ then $f(x) = f(y)$.

Solution: There exists a function f so that $x = y$ and $f(x) \neq f(y)$.

(b) Let A, B be sets. When is $(A \times B) \cap (B \times A)$ empty?

Solution: The intersection is empty when $A \cap B = \varnothing$.

(c) Define what it means for a set C to be well-ordered.

Solution: The set C is well ordered if every non-empty subset of C has a minimum element.

(d) Let $g : X \to Y$ be a function. Define precisely what it means for g to be surjective.

Solution: g is surjective when for all $y \in Y$ there exists $x \in X$ such that $g(x) = y$.

(e) Let $h : A \to B$ be a function. Define precisely what it means for h to be injective.

Solution: h is injective when for all $a_1, a_2 \in A$, $h(a_1) = h(a_2) \Rightarrow a_1 = a_2$. Equivalently $a_1 \neq a_2 \Rightarrow h(a_1) \neq h(a_2)$.

(f) Let $f : E \to F$ be a function and let $C \subseteq E$ and $D \subseteq F$.
Give precise definitions of the sets $f(C)$ and $f^{-1}(D)$.

Solution:

$$f(C) = \{f(x) \text{ s.t. } x \in C\} \qquad f^{-1}(D) = \{x \in E \text{ s.t. } f(x) \in D\}$$

5 marks 2. Let A, B, C be sets.

(a) Prove that if $A \subseteq B$ then $A - (B \cap C) \subseteq (A - C)$

Solution:

Proof. Assume $A \subseteq B$, and let $x \in A - (B \cap C)$. Hence $x \in A$ and $x \notin B \cap C$. Thus $x \notin B$ or $x \notin C$. Since $x \in A$, our assumption implies that $x \in B$, and so it must be the case that $x \notin C$. Hence $x \in A - C$. □

(b) Disprove that $(A - B) - (B - C) = (A - B) - C$.

Solution: Let $A = C = \{1\}$ and $B = \varnothing$. Then

- $A - B = A$ and $B - C = \varnothing$. Hence $LHS = A = \{1\}$.
- Now $A - B = A = \{1\}$ and $C = \{1\}$, so $RHS = \varnothing$.

Hence $LHS \neq RHS$.

5 marks 3. Use induction to prove that $81 \Big| (10^{n+1} - 9n - 10)$ for every non-negative integer n.

Solution:

Proof. We prove the statement by mathematical induction.

- Base case. When $n = 0$ we have $(10 - 0 - 10) = 0$ and so $81 | (10^1 - 9 \cdot 0 - 10)$ as required.
- Inductive step. Assume that $81 | (10^{k+1} - 9k - 10)$. Hence there is some $q \in \mathbb{Z}$ so that

  $$10^{k+1} - 9k - 10 = 81q$$

  Now we must show the case $n = k + 1$ and to do so we consider $810q$.

  $$810q = 10^{k+2} - 90k - 100$$

  and now consider

  $$\begin{aligned} 10^{k+2} - 9(k+1) - 10 &= (810q + 90k + 100) - 9k - 9 - 10 \\ &= 810q + 81k - 81 = 81(10q + k - 1) \end{aligned}$$

  Since $10q + k - 1 \in \mathbb{Z}$, we have that $81 | (10^{k+2} - 9(k+1) - 10)$ as required.

By mathematical induction, the statement is true for all non-negative integers.

□

6 marks 4. Let $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be defined by $f(x) = 1 - \dfrac{1}{x^2}$.

(a) Show that f is not injective.

Solution: Since $(-1)^2 = 1 = (1^2)$, it follows that $f(1) = 0 = f(-1)$. Hence f cannot be injective.

(b) Show that f is not surjective.

Solution: Consider $y = 2 \in \mathbb{R}$ — we show that there is no $x \in \mathbb{R} - \{0\}$ so that $f(x) = 2$.

$$\begin{aligned} 2 &= f(x) = 1 - \frac{1}{x^2} \\ 1 &= -\frac{1}{x^2} \\ x^2 + 1 &= 0 \end{aligned}$$

Since $x^2 \geq 0$ for any real x, it follows that this equation has no real solutions. Thus f is not surjective.

(c) Now define

$$g : \{x \in \mathbb{R} | x > 0\} \to \{x \in \mathbb{R} \text{ s.t. } x < 1\} \qquad g(x) = 1 - \frac{1}{x^2}$$

Show that g is bijective.

Solution: We show that it is injective and surjective.

- Injective. Let $x, z \in \{x \in \mathbb{R} | x > 0\}$ and assume $g(x) = g(z)$. Then

$$\begin{aligned} g(x) &= g(z) \\ 1 - x^{-2} &= 1 - z^{-2} \\ x^2 - z^2 &= 0 \\ (x - z)(x + z) &= 0 \end{aligned}$$

  Hence either $x = z$ or $x = -z$. Since both $x, z > 0$, the second is not possible. Thus $x = z$ and g is injective.

- Surjective. Let $y \in \{x \in \mathbb{R} \text{ s.t. } x < 1\}$. Then pick $x = \frac{1}{\sqrt{1-y}}$. Since $y < 1$, $1 - y > 0$ and so $\sqrt{1 - y} > 0$ and $x > 0$ as required. Then

$$\begin{aligned} g(x) &= 1 - \left(\frac{1}{\frac{1}{\sqrt{1-y}}} \right)^2 \\ &= 1 - (1 - y) = y \end{aligned}$$

  as required. And so g is surjective.

Thus g is bijective.

7 marks 5. Decide whether the following are true or false. Prove your answers

(a) Let A, B, C be sets. Then

$$(A \times C) - (B \times C) = (A - B) \times C.$$

Solution:

Proof. We must show both that $LHS \subseteq RHS$ and $RHS \subseteq LHS$.

- Let $(x, y) \in (A \times C) - (B \times C)$. Hence $x \in A$ and $y \in C$. Additionally $(x, y) \notin (B \times C)$, and thus either $x \notin B$ or $y \notin C$. Since we already know that $y \in C$, it must be the case that $x \notin B$. Hence $x \in (A - B)$ and thus $(x, y) \in (A - B) \times C$.
- Now let $(x, y) \in (A - B) \times C$. Thus $y \in C$ and $x \in A$ and $x \notin B$. Thus we must have $(x, y) \in A \times C$, and since $x \notin B$, it follows that $(x, y) \notin B \times C$. Hence $(x, y) \in (A \times C) - (B \times C)$.

□

(b) Let $n \in \mathbb{N}$, then $3^n > n^3$.

Solution: This is false. Consider $n = 3$, Then $LHS = 3^3 = 27$ and $RHS = 3^3 = 27$. But $27 \not> 27$.

(c) Let $f : A \to B$ be a surjective function and let $D \subseteq B$. Then

$$D \subseteq f(f^{-1}(D))$$

Solution: Let $y \in D$. Then since f is surjective, there is some $x \in A$ so that $f(x) = y$. Hence $x \in f^{-1}(D)$ (by definition). And it follows that $f(x) \in f(f^{-1}(D))$, Thus $f(x) = y \in f(f^{-1}(D))$ as required.