DIAGONALIZATION - PART 02

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In the previous lecture we found conditions to determine if a square matrix is diagonalizable. In this lecture we put everything together and give an algorithm to determine if a matrix is diagonalizable, and if it is, how to obtain the similar diagonal matrix.

1. Algorithm

Let A be an $n \times n$ matrix.

(1) Factor the characteristic polynomial of A as a product of linear factors
$$\det(A - tI) = (-1)^n (t - \lambda_1) \cdots (t - \lambda_n),$$
so we can find the eigenvalues of A and their multiplicities.

(2) For each distinct eigenvalue λ_i, obtain a basis for E_{λ_i}.

(3) A will be diagonalizable if for each distinct eigenvalue λ_i,
$$\dim(\mathrm{E}_{\lambda_i}) = m_i$$
where m_i is the multiplicity of λ_i.

(4) If A is diagonalizable we have that
$$A = Q\,D\,Q^{-1},$$
In this case:

- D is the diagonal matrix whose entries in the diagonal are the eigenvalues of A, where the eigenvalue λ_i will appears m_i times.
- Q is the matrix whose columns are the eigenvectors that form a basis for the eigenspaces E_{λ_i}, placed <u>in the same order as their corresponding eigenvectors were placed in D</u>.

2. Examples

Example 1. Consider the matrix $A = \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}$. We saw that

$$\det(A - tI) = (t-7)(t+2),$$

So we have

Eigenvalue	Multiplicity	$\dim(E_\lambda)$
7	1	
-2	1	

Since $1 \le \dim(E_\lambda) \le 1$ for $\lambda = 7, -2$, we have $\dim(E_\lambda) = 1$. We conclude

Eigenvalue	Multiplicity	$\dim(E_\lambda)$
7	1	1
-2	1	1

Thus, A is diagonalizable. We obtain a basis for E_7:

$$(A - 7\,I) = \left(\begin{array}{cc|c} 2-7 & 4 & 0 \\ 5 & 3-7 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -4/5 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

so $\left\{\begin{pmatrix} 4/5 \\ 1 \end{pmatrix}\right\}$ is a basis for E_7. Notice that we can take $\left\{\begin{pmatrix} 4 \\ 5 \end{pmatrix}\right\}$ as a basis instead of the previous one.

We obtain a basis for E_{-2}:

$$(A + 2\,I) = \left(\begin{array}{cc|c} 2+2 & 4 & 0 \\ 5 & 3+2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

so $\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$. is a basis for E_{-2}.

Thus, $A = Q\,D\,Q^{-1}$ reads:

$$\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 5 & 1 \end{pmatrix}^{-1}.$$

Example 2. Consider the matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. We saw that

$$\det(A - tI) = -(t-4)(t-3)^2.$$

So we have

Eigenvalue	Multiplicity	$\dim(E_\lambda)$
4	1	
3	2	

Since $1 \leq \dim(E_4) \leq 1$, we have that

$$\dim(E_4) = 1.$$

In order to find $\dim(E_3)$ we need to work with the system

$$(A - 3\,I) = \left(\begin{array}{ccc|c} 3-3 & 1 & 0 & 0 \\ 0 & 3-3 & 0 & 0 \\ 0 & 0 & 4-3 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since this matrix has rank two, we have

$$\dim(E_3) = 3 - 2 = 1$$

This gives us

Eigenvalue	Multiplicity	$\dim(E_\lambda)$
4	1	1
3	2	1

Thus,

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

is NOT diagonalizable.

Example 3. Determine if

$$A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

is diagonalizable.

Since

$$\det(A - tI) = -(t-5)(t-3)^2$$

we have

Eigenvalue	Multiplicity	$\dim(E_\lambda)$
5	1	
3	2	

We can check that

$$\mathrm{Null}(A - 5I) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

and

$$\mathrm{Null}(A - 3I) = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We conclude

Eigenvalue	Multiplicity	$\dim(E_\lambda)$
5	1	1
3	2	2

Hence A is diagonalizable and

$$\begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$$

3. Miscellaneous Results

Theorem 4. *Two matrices are similar if and only if they have the same eigenvalues.*

Theorem 5. *Let $\vec{x}$ be an eigenvector of A with eigenvalue λ. Let $l > 0$ be an integer. Then $\vec{x}$ is an eigenvector of A^l with eigenvalue λ^l.*

Theorem 6. *An $n \times n$ matrix is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.*

Recall that a matrix A is symmetric if $A^T = A$.

Theorem 7. *If A is a $n \times n$ symmetric matrix then A is diagonalizable. Moreover, the eigenvectors that form a basis for the distinct eigenspaces form an orthonormal basis of $\mathbb{R}^n$.*

Remark 8. Given a diagonalizable matrix A, notice that the expression $A = Q\,D\,Q^{-1}$ obtained by the method above is not unique. More precisely, it depends on:

- The order on which you place the eigenvalues on D.
- The base you are considering for the eigenspaces E_λ.

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