DETERMINANTS

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In this lecture we will see how we can determine, without using a system of equations, if a collection of n vectors in $\mathbb{R}^n$ is linearly independent. The idea is to associate to any square matrix a number, the determinant, which will contain such information.

1. Definition of Determinant

Given a square matrix $A = (a_{ij})$ of order n, the **submatrix** A_{ij} is obtained by deleting the i-th row and the j-th column of A.

$$\left(\begin{array}{ccc|c|ccc}
a_{11} & \cdots & a_{1\,j-1} & a_{1j} & a_{1\,j+1} & \cdots & a_{1n} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{i-1\,1} & \cdots & a_{i-1\,j-1} & a_{i-1\,j} & a_{i-1\,j+1} & \cdots & a_{i-1\,n} \\
\hline
a_{i1} & \cdots & a_{i\,j-1} & a_{ij} & a_{i\,j+1} & \cdots & a_{in} \\
\hline
a_{i+1} & \cdots & a_{i+1\,j-1} & a_{i+1\,j} & a_{i+1\,j+1} & \cdots & a_{i+1\,n} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1} & \cdots & a_{m\,j-1} & a_{mj} & a_{m\,j+1} & \cdots & a_{mn}
\end{array}\right)$$

Remark 1. If A is an $n \times n$ matrix, then any submatrix A_{ij} is an $(n-1) \times (n-1)$ matrix.

Example 2.

$$A = \begin{pmatrix} 2 & -3 & 10 \\ 0 & 5 & -1 \\ 7 & 9 & 4 \end{pmatrix}$$

Then

$$A_{21} = \begin{pmatrix} -3 & 10 \\ 9 & 4 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 0 & -1 \\ 7 & 4 \end{pmatrix}$$

$$A_{31} = \begin{pmatrix} -3 & 10 \\ 5 & -1 \end{pmatrix} \qquad A_{13} = \begin{pmatrix} 0 & 5 \\ 7 & 9 \end{pmatrix}$$

Definition 3. The **determinant** of a square $n \times n$ matrix A, denoted as $\det(A)$ or $\mid A \mid$, is defined recursively as follows:

(1) If $n = 1$, i.e., $A = (a_{11})$, then

$$\det(a_{11}) = a_{11}$$

(2) If $n = 2$, then

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

(3) If $n = 3$, then

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\ \det(A_{11}) - a_{12}\ \det(A_{12}) + a_{13}\ \det(A_{13})$$

(4) If $n > 3$, then

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= a_{11}\ \det(A_{11}) - a_{12}\ \det(A_{12}) + \cdots + (-1)^{n+1} a_{1n}\ \det(A_{1n})$$

where $\det(A_{1j})$ is defined in a recursive way using the formula given above.

Remark 4. Notice that as we move along the first row, the sign of a_{1i} alternates.

Example 5.

$$\det \begin{pmatrix} -1 & 3 \\ 2 & 7 \end{pmatrix} = (-1)(7) - (3)(2) = -13.$$

Example 6.

$$\det\begin{pmatrix} 1 & 3 & -3 \\ -3 & 5 & 2 \\ -4 & 4 & -6 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ & & \\ & & \end{pmatrix}$$

$$= 1\det\begin{pmatrix} 5 & 2 \\ 4 & -6 \end{pmatrix} - 3\det\begin{pmatrix} -3 & 2 \\ -4 & -6 \end{pmatrix} + (-3)\det\begin{pmatrix} -3 & 5 \\ -4 & 4 \end{pmatrix}$$

$$= 1(-30-8) - 3(18-(-8)) - 3(-12-(-20))$$

$$= -38 - 3(26) - 3(8) = -140.$$

Example 7.

$$\det\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix} \qquad \begin{pmatrix} + & - & + & - \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

$$= 2\det\begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} - 0\det\begin{pmatrix} 0 & 3 & -3 \\ -2 & -5 & 2 \\ 4 & 4 & -6 \end{pmatrix}$$

$$+ 0\det\begin{pmatrix} 0 & 1 & -3 \\ -2 & -3 & 2 \\ 4 & -4 & -6 \end{pmatrix} - 1\det\begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix}$$

$$= 2(40) - 0 + 0 - 1(48) = 32.$$

2. Computational Techniques, Part 1

Sometimes it is more convenient to compute the determinant of a matrix by moving, or expanding, along any row or column of the matrix, instead of moving along the first row. This is what the next theorem is about.

Theorem 8 (Cofactor Expansion)**.** *Given an $n \times n$ A, the determinant can be computed as follows:*

(1) *Let i be a fixed row number. Then*

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + (-1)^{i+2} a_{i2} \det(A_{i2}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in}).$$

(2) *Let j be a fixed column number. Then*

$$\det(A) = (-1)^{j+1} a_{1j} \det(A_{1j}) + (-1)^{j+2} a_{2j} \det(A_{2j}) + \cdots + (-1)^{j+n} a_{nj} \det(A_{nj}).$$

Remark 9. The signs $(-1)^{i+j}$ alternate as we move one spot either horizontally or vertically.

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & & & & \ddots \end{pmatrix}.$$

Example 10. By expanding along the second row:

$$\det\begin{pmatrix} 1 & 3 & -3 \\ -3 & 5 & 2 \\ -4 & 4 & -6 \end{pmatrix} \qquad \begin{pmatrix} & & \\ - & + & - \\ & & \end{pmatrix}$$

$$= -(-3)\det\begin{pmatrix} 3 & -3 \\ 4 & -6 \end{pmatrix} + (5)\det\begin{pmatrix} 1 & -3 \\ -4 & -6 \end{pmatrix} - (2)\det\begin{pmatrix} 1 & 3 \\ -4 & 4 \end{pmatrix}$$

$$= -(-3)(-18+12) \;+\; (5)(-6-12) \;-\; (2)(4+12)$$

$$= -18 - 90 - 32 = -140.$$

Remark 11. The method of evaluating a determinant by expanding along a row or column is more efficient when it is applied to the row or column with the largest number of zeros.

Example 12. By expanding along the first column we get

$$\det\begin{pmatrix} -1 & -5 & -6 \\ 0 & -13 & -11 \\ 0 & 40 & 26 \end{pmatrix} \;=\; (-1)\,[(-13)(26) - (-11)(40)]$$

$$= \;-102.$$

3. Properties of Determinants

The following result summarizes the basic properties of determinants.

Theorem 13. *Let A be a square matrix.*

(1) *If A has a zero row or column, then* $\det(A) = 0$.
(2) *If A has two proportional rows (or two proportional columns), then* $\det(A) = 0$.
(3) *If A is an upper (lower) triangular matrix, then* $\det(A)$ *is the product of its diagonal components.*
(4) *If A is a diagonal matrix, then* $\det(A)$ *is the product of its diagonal components.*
(5) $\det(I_n) = 1$.
(6) $\det(A) = \det(A^T)$.

Example 14. Some examples are:

$$\det\begin{pmatrix} 7 & 4 & \mathbf{0} & 1 \\ -1 & 3 & \mathbf{0} & 4 \\ 9 & 2 & \mathbf{0} & -9 \\ 7 & 1 & \mathbf{0} & 10 \end{pmatrix} = 0, \qquad \det\begin{pmatrix} \mathbf{7} & 4 & 4 & 1 \\ 0 & \mathbf{3} & 8 & 7 \\ 0 & 0 & \mathbf{-2} & 4 \\ 0 & 0 & 0 & \mathbf{10} \end{pmatrix} = -420.$$

Theorem 15 (Main Theorem)**.** (1) *A square matrix A is invertible if and only if*

$$\det(A) \neq 0.$$

(2) *If A and B are two $n \times n$ matrices, then*

$$\det(AB) = \det(A)\,\det(B).$$

Remark 16. Some consequences of the theorem above are the following.

Let A be an $n \times n$ matrix. Then

- $\det(A^n) = (\det(A))^n$.
- $\det\left(A^{-1}\right) = \det(A)^{-1} = \dfrac{1}{\det(A)}$.

Example 17. Assume that A and B are $n \times n$ matrices such that $\det(A) = 100$, and $\det(B) = 5$. Then

$$\det\left(\left(A^T\right)\left(B^{-1}\right)\right) = \det\left(A^T\right)\det\left(B^{-1}\right) = \det(A)\frac{1}{\det(B)} = \frac{100}{5} = 20.$$

4. Computational Techniques, Part 2

Theorem 18. *Let A be an $n \times n$ matrix.*

(1) *If B is the matrix obtained by interchanging any two rows of A, then*
$$\det(B) = -\det(A).$$

(2) *If B is the matrix obtained by multiplying a* **single** *row of A by any non-zero scalar α, then*
$$\det(B) = \alpha \det(A).$$

(3) *If B is the matrix obtained by adding a multiple of one row of A to another row of A, then*
$$\det(B) = \det(A).$$

Caution: Notice that in (2) from the Theorem above the multiplier α is applied to a **single row only**.

This implies that for an $n \times n$ matrix A, regarding αA as the scalar product for matrices by a non-zero scalar α, we get

$$\det(\alpha A) = \alpha^n \det(A).$$

Example 19. Assume that A and B are $n \times n$ matrices such that $\det(A) = 100$, and $\det(B) = 5$. Then

$$\begin{aligned}\det\left(\left(A^T\right)\left(2B^{-1}\right)\right) &= \det\left(A^T\right)\det\left(2B^{-1}\right) = \det(A)\frac{2^n}{\det(B)} \\ &= 100\frac{2^n}{5} = (20)2^n.\end{aligned}$$

Remark 20. Since $\det(A) = \det(A^T)$, we can replace the word **row** in the previous theorem and get similar statements for the respective **column operations**.

Example 21. If $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 10$, then

$$\det\begin{pmatrix} f & e & d \\ c & b & a \\ i & h & g \end{pmatrix} = 10, \qquad \det\begin{pmatrix} b & a & 2c \\ e & d & 2f \\ h & g & 2i \end{pmatrix} = -20,$$

4.1. **A Method for Computing the Determinant.** Using elementary row (or column) operations obtain a column of the form $\begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix}$ (or a row of the form $\begin{matrix} 1 & 0 & \cdots & 0 \end{matrix}$) and compute the determinant by expanding along this column (or row).

Example 22.

$$\begin{aligned}
\det\begin{pmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 6 & 1 & 2 \end{pmatrix} &= \det\begin{pmatrix} 2 & 1 & 1 & 5 \\ -1 & 0 & -5 & -6 \\ 2 & 0 & -3 & 1 \\ -9 & 0 & -5 & -28 \end{pmatrix} \\
&= -(1)\det\begin{pmatrix} -1 & -5 & -6 \\ 2 & -3 & 1 \\ -9 & -5 & -28 \end{pmatrix} \\
&= -(1)\det\begin{pmatrix} -1 & -5 & -6 \\ 0 & -13 & -11 \\ 0 & 40 & 26 \end{pmatrix} \\
&= (-1)(-1)\left[(-13)(26) - (-11)(40)\right] \\
&= 102.
\end{aligned}$$

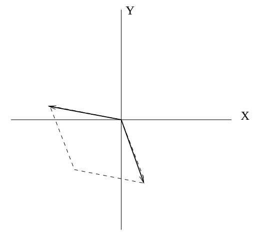
4.2. **Another Method for Computing the Determinant.** Using elementary row operations transform any matrix into an upper triangulated matrix and compute the determinant keeping track of the operations done.

Example 23.

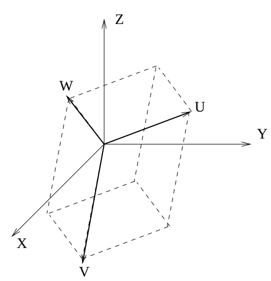
$$
\begin{aligned}
\det \begin{pmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 6 & 1 & 2 \end{pmatrix} &= -\det \begin{pmatrix} 1 & 1 & -4 & -1 \\ 2 & 1 & 1 & 5 \\ 2 & 0 & -3 & 1 \\ 3 & 6 & 1 & 2 \end{pmatrix} \\
&= -\det \begin{pmatrix} 1 & 1 & -4 & -1 \\ 0 & -1 & 9 & 7 \\ 0 & -2 & 5 & 3 \\ 0 & 3 & 13 & 5 \end{pmatrix} \\
&= -\det \begin{pmatrix} 1 & 1 & -4 & -1 \\ 0 & -1 & 9 & 7 \\ 0 & 0 & -13 & -11 \\ 0 & 0 & 40 & 26 \end{pmatrix} \\
&= -(-13)\det \begin{pmatrix} 1 & 1 & -4 & -1 \\ 0 & -1 & 9 & 7 \\ 0 & 0 & 1 & 11/13 \\ 0 & 0 & 40 & 26 \end{pmatrix} \\
&= -(-13)\det \begin{pmatrix} 1 & 1 & -4 & -1 \\ 0 & -1 & 9 & 7 \\ 0 & 0 & 1 & 11/13 \\ 0 & 0 & 0 & -102/13 \end{pmatrix} \\
&= 13\left(-\frac{102}{13}\right)(1)(-1)(1) \\
&= 102.
\end{aligned}
$$

5. Geometric Interpretation of Determinant

(1) Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$. Consider the matrix A whose first column is given by $\vec{u}$ and second column is given by $\vec{v}$. Then the area of the parallelogram having the vectors $\vec{u}$ and $\vec{v}$ as adjacent sides is given by $|\det(A)|$.



(2) Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$. Consider the matrix A whose first column is given by $\vec{u}$, second column is given by $\vec{v}$, and third column is $\vec{w}$. Then the volume of the parallelepiped having the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ as adjacent sides is given by $|\det(A)|$.



Example 24. The volume of the parallelepiped determine by the vectors $\vec{v}_1 = (-1, 2, 0)$, $\vec{v}_2 = (1, 0, 4)$ and $\vec{v}_3 = (1, -2, 2)$ is equal to

$$\det \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & -2 \\ 0 & 4 & 2 \end{pmatrix} = -4$$

Thus the volume of the parallelepiped is 4.

6. Applications to Linear Systems

Theorem 25. *If A is an $n \times n$ matrix then the following are equivalent:*

(1) *A is invertible.*

(2) *The RREF of A is I_n.*

(3) $\operatorname{rank}(A) = n$.

(4) $\operatorname{Col}(A) = \mathbb{R}^n$.

(5) $\operatorname{Row}(A) = \mathbb{R}^n$.

(6) *$A\vec{x} = \vec{0}$ has only the trivial solution.*

(7) *$A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^n$.*

(8) *$A\vec{x} = \vec{b}$ has exactly one solution for every $\vec{b}$ in $\mathbb{R}^n$.*

(9) $\det(A) \neq 0$.

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