ORTHOGONAL AND ORTHONORMAL BASES

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Among all the bases $\mathbb{R}^n$ has, the standard basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ has the property that for any vector

$$\vec{v} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

we have

$$\vec{v} = a_1\vec{e}_1 + \cdots + a_n\vec{e}_n.$$

An analogous easy description of a vector $\vec{v}$ in terms of a basis only happens for a special type of bases of $\mathbb{R}^n$. It becomes clear what type of bases we are looking for once we realize that

$$\text{proj}_{\vec{e}_i}(\vec{v}) = \frac{\vec{e}_i \bullet \vec{v}}{\|\vec{e}_i\|^2}\, \vec{e}_i = a_i\vec{e}_i.$$

In other words, the coefficients describing $\vec{v}$ as a linear combination of $\{\vec{e}_1, \ldots, \vec{e}_n\}$ are of the form

$$\frac{\vec{e}_i \bullet \vec{v}}{\|\vec{e}_i\|^2} = \vec{e}_i \bullet \vec{v},$$

and

$$\vec{v} = \text{proj}_{\vec{e}_1}(\vec{v}) + \cdots + \text{proj}_{\vec{e}_n}(\vec{v}).$$

Notice that the properties of the basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ used here were that they are orthogonal to each other and that their norm is 1. These are the properties that a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ should have, if we want to obtain

$$\vec{v} = (\vec{v}_i \bullet v)\ \vec{v}_1 + \cdots + (\vec{v}_n \bullet v)\ \vec{v}_n$$

This lecture is about describing such type of bases for $\mathbb{R}^n$. Before we proceed, lets recall what is the norm of a vector and the projection of a vector along another vector.

0.0.1. The **norm** of a vector $\vec{v}$ in $\mathbb{R}^n$ was defined as

$$\|\vec{v}\| = \left\| \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \right\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

The properties of the norm we should keep in mind are

1. $\|\vec{v}\| \geq 0$, and $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$.
2. $\|\alpha\vec{v}\| = |\alpha| \, \|\vec{v}\|$, for any scalar α.
3. $\|\vec{v}\|^2 = \vec{v} \bullet \vec{v}$.

0.0.2. Let $\vec{v}$ and $\vec{u}$ be two vectors in $\mathbb{R}^n$, with $\vec{u} \neq \vec{0}$. Then $\mathrm{proj}_{\vec{u}}(\vec{v})$ was defined as

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\vec{u} = \frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2}\vec{u}.$$

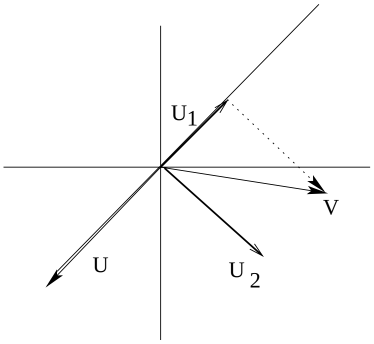
Remark 1. (1) If $\vec{u}$ and $\vec{v}$ are orthogonal, then

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{0}{\|\vec{u}\|^2}\vec{u} = \vec{0}.$$

(2) If $\alpha\vec{u} = \vec{v}$, $\alpha \neq 0$, then

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2}\vec{u} = \alpha\vec{u}$$

(3) Set $\vec{u}_1 = \mathrm{proj}_{\vec{u}}(\vec{v})$, and $\vec{u}_2 = \vec{v} - \mathrm{proj}_{\vec{u}}(\vec{v})$, so we have the following picture



Notice that the vectors $\vec{u}_1$ and $\vec{u}_2$ were defined so that

$$\vec{u}_1 \bullet \vec{u}_2 = 0 \qquad \text{and} \qquad \vec{v} = \vec{u}_1 + \vec{u}_2.$$

1. ORTHOGONAL BASES

Definition 2. Let $B = \{\vec{v}_1, \ldots, \vec{v}_r\}$ be a basis for a subspace $V \subseteq \mathbb{R}^n$. We say that B is an *orthogonal basis* for V if

$$\vec{v}_i \bullet \vec{v}_j = 0,$$

for any $1 \leq i \neq j \leq r$.

Example 3. The set $S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^2$. Indeed, this follows from

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (1)(-1) + (1)(1) = 0.$$

Example 4. The set $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. Indeed, this follows from

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = (1)(1) + (1)(-1) + (0)(1) = 0,$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = (1)(-1) + (1)(1) + (0)(2) = 0,$$

$$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = (-1)(1) + (1)(-1) + (2)(1) = 0.$$

Example 5. The set $S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ is NOT an orthogonal basis for $\mathbb{R}^2$. Indeed, this follows from

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (1)(1) + (1)(2) = 3 \neq 0.$$

2. MAIN RESULTS

Theorem 6. *Let* $S = \{\vec{v}_1, \ldots, \vec{v}_m\}$ *be a set of non-zero vectors such that* $\vec{v}_i \bullet \vec{v}_j = 0$*, for any* $1 \le i \ne j \le m$*. Then* S *is a linearly independent set.*

Proof. Let $\alpha_1, \ldots, \alpha_m$ be scalars such that

$$\vec{0} = \alpha_1 \vec{v}_1 + \cdots + \alpha_m \vec{v}_m.$$

Then

$$0 = \vec{0} \bullet \vec{v}_1 = \alpha_1 (\vec{v}_1 \bullet \vec{v}_1) + \cdots + \alpha_1 (\vec{v}_1 \bullet \vec{v}_m) = \alpha_1 (\vec{v}_1 \bullet \vec{v}_1).$$

Since $\vec{v}_1 \ne \vec{0}$ we have that $\vec{v}_1 \bullet \vec{v}_1 > 0$, so we conclude that $\alpha_1 = 0$.

We keep repeating this process with the remaining vectors in S so that we get $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$. Q.E.D.

Theorem 7. *Let* $S = \{\vec{v}_1, \ldots, \vec{v}_r\}$ *be an orthogonal basis for a subspace* $V \subseteq \mathbb{R}^n$*. Then any vector* $\vec{w}$ *in* V *can be written as*

$$\begin{aligned} \vec{w} &= \mathrm{proj}_{\vec{v}_1}(\vec{w}) + \cdots + \mathrm{proj}_{\vec{v}_r}(\vec{w}) \\ &= \frac{\vec{v}_1 \bullet \vec{w}}{\|\vec{v}_1\|} \vec{v}_1 + \cdots + \frac{\vec{v}_r \bullet \vec{w}}{\|\vec{v}_r\|} \vec{v}_r \end{aligned}$$

Proof. Since $\vec{v}$ is in V we have that there are scalars $\alpha_1, \ldots, \alpha_r$ such that

$$\vec{v} = \alpha_1 \vec{v}_1 + \cdots + \alpha_r \vec{v}_r.$$

Then, for any $1 \le i \le r$ we have

$$\begin{aligned} \vec{v} \bullet \vec{v}_i &= (\alpha_1 \vec{v}_1 + \cdots + \alpha_r \vec{v}_r) \bullet \vec{v}_i \\ &= \alpha_i (\vec{v}_i \bullet \vec{v}_i) \\ &= \alpha_i \|\vec{v}_i\|^2. \end{aligned}$$

Hence, $\alpha_i = \dfrac{\vec{v} \bullet \vec{v}_i}{\|\vec{v}_i\|^2}$. Q.E.D.

Note 8. Suppose $U \subset \mathbb{R}^n$ is a subspace, with $U \neq \mathbb{R}^n$. Consider an orthogonal basis $\{\vec{u}_1, \ldots, \vec{u}_d\}$ of U. If $\vec{v} \in \mathbb{R}^n$ is a vector not in U, we still can consider the vector

$$\text{proj}_U(\vec{v}) = \text{proj}_{\vec{u}_1}(\vec{v}) + \cdots + \text{proj}_{\vec{u}_d}(\vec{v}).$$

The vector $\text{proj}_U(\vec{v})$ sometimes is called the **orthogonal projection** of $\vec{v}$ on U. This vector is the closest vector to $\vec{v}$ in U, and that is why sometimes can be also called **the best approximation to $\vec{v}$ in U**.

Theorem 9. *Any subspace V of $\mathbb{R}^n$ has an orthogonal basis.*

The proof of this theorem consists of constructing such orthogonal basis using the Gram-Schmidt Orthogonalization Process.

3. GRAM-SCHMIDT ORTHOGONALIZATION PROCESS

Theorem 10 (Gram-Schmidt Orthogonalization Process)**.** *Assume that $S = \{\vec{v}_1, \ldots, \vec{v}_r\}$ is a set of linearly independent vectors. Then the vectors*

$$\begin{aligned}
\vec{u}_1 &:= \vec{v}_1 \\[2em]
\vec{u}_2 &:= \vec{v}_2 - \text{proj}_{\vec{u}_1}(\vec{v}_2) \\[2em]
\vec{u}_3 &:= \vec{v}_3 - \text{proj}_{\vec{u}_1}(\vec{v}_3) - \text{proj}_{\vec{u}_2}(\vec{v}_3) \\
&\vdots \\
\vec{u}_r &:= \vec{v}_r - \text{proj}_{\vec{u}_1}(\vec{v}_r) - \cdots - \text{proj}_{\vec{u}_{r-1}}(\vec{v}_r)
\end{aligned}$$

define an orthogonal basis for $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_r\}$.

Example 11. Obtain an orthogonal basis for

$$V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

First, notice that the vectors generating V are linearly independent.

Then, applying Gram-Schmidt process we obtain:

(1) $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$(2)\ \vec{u}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$(3)\ \vec{u}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \tfrac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

Thus,

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

is an orthogonal basis for V.

Remark 12. Notice that $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$ is also an orthogonal basis for the vector space V from the previous example.

Example 13. Using the result from the previous example, in this example we will illustrate the statement of theorem 7.

Let $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ be a vector in $\mathbb{R}^3$. Since

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

is an orthogonal basis for $\mathbb{R}^3$, theorem 7 says that

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \text{proj}_{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \quad + \quad \text{proj}_{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \quad + \quad \text{proj}_{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}}{\left\| \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\|^2} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad + \quad \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad + \quad \frac{2}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

4. ORTHONORMAL BASES

Definition 14. An orthogonal basis $B = \{\vec{v}_1, \ldots, \vec{v}_r\}$ for a subspace $V \subseteq \mathbb{R}^n$ is called an *orthonormal basis* for V if $\|\vec{v}_i\| = 1$, for any $1 \leq i \leq r$.

Example 15. The set $S = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^2$.

Indeed, this follows from

$$\left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \bullet \left(\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) = \frac{1}{2} \left((1)(-1) + (1)(1) \right) = \frac{0}{2} = 0.$$

and

$$\left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \frac{\sqrt{2}}{\sqrt{2}} = 1,$$

$$\left\| \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\| = \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\| = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

Example 16. The set $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$, but NOT an orthonormal basis for $\mathbb{R}^3$.

Indeed, we already saw that S is an orthogonal basis for $\mathbb{R}^3$, but it is not orthonormal since

$$\left\| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\| = \sqrt{2} \neq 1.$$

Recall that given a vector $\vec{v}$, we can always obtain a vector $\vec{v}'$ with the same direction that $\vec{v}$ but with norm 1:

$$\vec{v}' = \frac{\vec{v}}{\|\vec{v}\|}.$$

Such vector $\vec{v}'$ is called the **normalization** of $\vec{v}$.

Theorem 17. *Any subspace $V \subseteq \mathbb{R}^n$ has an orthonormal basis.*

Proof. From Theorem 9 any subspace V has an orthogonal basis $\{\vec{u}_1, \ldots, \vec{u}_r\}$. Then the set of vectors

$$\left\{ \frac{\vec{u}_1}{\|\vec{u}_1\|}, \ldots, \frac{\vec{u}_r}{\|\vec{u}_r\|} \right\}$$

is an orthonormal basis for V. Q.E.D.

Example 18. We saw that

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

is an orthogonal basis for $\mathbb{R}^3$. Then the orthonormal basis for $\mathbb{R}^3$ induced by S is

$$\left\{ \frac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\|^2}, \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\|^2}, \frac{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}{\left\| \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\|^2} \right\}$$

$$= \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

Example 19. Obtain an orthonormal basis for

$$V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \right\}$$

First notice that the three vectors generating V are linearly independent. Next, we obtain an orthogonal basis:

(1) $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

(2) $\vec{u}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} - \dfrac{\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix}$

(3) $\vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} - \dfrac{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \dfrac{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\|^2} \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -4 \\ -2 \\ -1 \\ 6 \end{pmatrix}$

Thus, $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{7} \begin{pmatrix} -4 \\ -2 \\ -1 \\ 6 \end{pmatrix} \right\}$ is an orthogonal basis for V.

Finally, we obtain an orthonormal basis by computing the normalizations of the vectors in S:

$$(1)\ \frac{\begin{pmatrix}1\\1\\0\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right\|} \quad = \quad \tfrac{1}{3}\begin{pmatrix}1\\1\\0\\1\end{pmatrix},$$

$$(2)\ \frac{\tfrac{1}{3}\begin{pmatrix}4\\-5\\0\\1\end{pmatrix}}{\left\|\tfrac{1}{3}\begin{pmatrix}4\\-5\\0\\1\end{pmatrix}\right\|} \quad = \quad \frac{\tfrac{1}{3}\begin{pmatrix}4\\-5\\0\\1\end{pmatrix}}{\tfrac{1}{3}\left\|\begin{pmatrix}4\\-5\\0\\1\end{pmatrix}\right\|} \quad = \quad \frac{\begin{pmatrix}4\\-5\\0\\1\end{pmatrix}}{\left\|\begin{pmatrix}4\\-5\\0\\1\end{pmatrix}\right\|} \quad = \quad \tfrac{1}{42}\begin{pmatrix}4\\-5\\0\\1\end{pmatrix}$$

$$(3)\ \frac{\tfrac{1}{7}\begin{pmatrix}-4\\-2\\-1\\6\end{pmatrix}}{\left\|\tfrac{1}{7}\begin{pmatrix}-4\\-2\\-1\\6\end{pmatrix}\right\|} \quad = \quad \frac{\tfrac{1}{7}\begin{pmatrix}-4\\-2\\-1\\6\end{pmatrix}}{\tfrac{1}{7}\left\|\begin{pmatrix}-4\\-2\\-1\\6\end{pmatrix}\right\|} \quad = \quad \frac{\begin{pmatrix}-4\\-2\\-1\\6\end{pmatrix}}{\left\|\begin{pmatrix}-4\\-2\\-1\\6\end{pmatrix}\right\|} \quad = \quad \tfrac{1}{57}\begin{pmatrix}-4\\-2\\-1\\6\end{pmatrix}$$

Thus, $\left\{\tfrac{1}{3}\begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \tfrac{1}{42}\begin{pmatrix}4\\-5\\0\\1\end{pmatrix}, \tfrac{1}{57}\begin{pmatrix}-4\\-2\\-1\\6\end{pmatrix}\right\}$ is an orthonormal basis for V.

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