ORTHOGONAL SPACES

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Now we will get different descriptions of $\mathbb{R}^n$. The first case is in terms of orthogonal subspaces.

1. Orthogonal Complement

Definition 1. Let V be a subspace of $\mathbb{R}^n$. The *orthogonal complement of* V, denoted as $V^\perp$, is the set of all the vectors in $\mathbb{R}^n$ that are orthogonal to all the vectors in V.

In other words, for any $\vec{v}$ in V, and any $\vec{w}$ in $V^\perp$, we have $\vec{v} \bullet \vec{w} = 0$.

Remark 2. Notice that if $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is a basis for V, then $\vec{w}$ is in $V^\perp$ if and only if $\vec{w} \bullet \vec{v}_i = 0$, for $1 \leq i \leq r$.

Example 3. If A is any $m \times n$ matrix, and $V = \text{Row}(A)$, we have by definition that $V^\perp = (\text{Row}(A))^\perp = \text{Null}(A)$.

Example 4. From the definition of the transpose of a matrix we know that $\text{Col}(A) = \text{Row}(A^T)$. Thus, we have

$$\text{Col}\,(A)^\perp = \text{Row}\left(A^T\right)^\perp = \left(\text{Null}\left(A^T\right)\right).$$

The following theorem summarizes all we have to say about orthogonal complements.

Theorem 5. *Let $\{\vec{v}_1, \ldots, \vec{v}_r\}$ be a basis for a subspace V of $\mathbb{R}^n$. Then*

(1) *$V^\perp$ is a subspace of $\mathbb{R}^n$ and $V \cap V^\perp = \{\vec{0}\}$.*
(2) *$\vec{v} \bullet \vec{w} = 0$ for any $\vec{v}$ in V and any $\vec{w}$ in $V^\perp$.*
(3) *If $\{\vec{w}_1, \ldots, \vec{w}_{n-r}\}$ is a basis for $V^\perp$, then $\{\vec{v}_1, \ldots, \vec{v}_r, \vec{w}_1, \ldots, \vec{w}_{n-r}\}$ is a basis for $\mathbb{R}^n$. In particular, any vector $\vec{u}$ in $\mathbb{R}^n$ can be written as*
$$\vec{u} = \vec{v}_u + \vec{w}_u,$$
where $\vec{v}_u$ is a vector in V and $\vec{w}_u$ is a vector in $V^\perp$.

Example 6. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -1 & 6 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

We want to find a basis of Row(A) and Null(A). In order to do that we apply Gaussian elimination to A:

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -1 & 6 \\ 0 & 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

- We have that $3 = \text{rank}(A) = \dim \text{Row}(A)$, and that a basis for Row(A) is:

$$\text{Row}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

- From the Dimension Theorem we must have

$$\dim(\text{Null}(A)) = n - \text{rank}(A) = 4 - 3 = 1.$$

  And this is verified when we solved the homogeneous system $A\vec{x} = \vec{0}$:

$$x_1 = x_4, \qquad x_2 = -2x_4, \qquad x_3 = 0.$$

  Set $x_4 = t$. Then

$$\text{Null}(A) = \left\{ \begin{pmatrix} t \\ -2t \\ 0 \\ t \end{pmatrix} \mid t \in \mathbb{R}^n \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

- With a direct computation we can verified that any vector in the basis for $\text{Row}(A)$ is orthogonal to the basis in $\text{Null}(A)$. Thus,

$$(\text{Row}(A))^{\perp} = \text{Null}(A).$$

 Or equivalently,

$$(\text{Null}(A))^{\perp} = \text{Row}(A).$$

- Finally, notice that

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

 is a linearly independent set, so it is a basis for $\mathbb{R}^4$.

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