INVERTIBLE MATRICES

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One good feature of square matrices is that we have the notion of inverse matrices. In particular, we have that the square matrices behave as numbers. A consequence of this is that we can recover a version of the cancellation property.

1. Inverse of a Matrix

Definition 1. Let A be a square matrix. We say that A is **invertible** if there is another square matrix B such that

$$AB = BA = I$$

In such case we say that B is the **inverse** of A.

Notice that if A is invertible with inverse B, then B is also invertible with inverse A. We will denote the inverse matrix of A by A^{-1}.

The inverse of a diagonal matrix is easy to obtain:

Example 2. The inverse of the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 10 \end{pmatrix}$ is given by

$$A^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/10 \end{pmatrix}.$$

In general, this is not the case.

Example 3. The inverse of the matrix $A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ is given by

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}.$$

Remark 4. (1) Only square matrices might have an inverse.

(2) Not every square matrix will have an inverse.

(3) Assume that A is an invertible matrix. If $AC = AB$ then $B = C$. Similarly, if $CA = BA$ then $C = B$.

Theorem 5. *If the inverse of a matrix exists, then the inverse is unique.*

Proof. Let A be an $n \times n$ invertible matrix. Let B and B' inverse matrices of A. Then

$$B = BI_n = B(AB') = (BA)B' = I_nB' = B'.$$

Q.E.D.

Theorem 6. *An $n \times n$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse if and only if $ad - bc \neq 0$. In this case,*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof. A matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if and only if there is a matrix

$$B = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

such that

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

But this means that the vectors

$$\begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix}$$

are solutions of the systems

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

respectively.

But we know how to solve these two systems simultaneously by means of Gaussian elimination:

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & d/D & -b/D \\ 0 & 1 & -c/D & a/D \end{array}\right)$$

where the computations will show that the two matrices above are equivalent if and only if $D = \frac{1}{ad-bc} \neq 0$. Q.E.D.

Example 7. Consider the matrix $A = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$.

Since

$$(-2)(1) - (1)(1) = -2 - 1 = -3 \neq 0,$$

we have that

$$A^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}.$$

Example 8. Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$.

Since

$$(1)(-4) - (2)(-2) = -4 - (-4) = 0,$$

we have that A has no inverse.

Theorem 9. *Let A and B be two invertible matrices. Then*

(1) *AB is invertible with inverse $B^{-1}A^{-1}$.*

(2) $\left(A^{-1}\right)^{-1} = A$.

(3) *A^n is invertible with inverse $\left(A^{-1}\right)^n$.*

We will use the notation $\left(A^{-1}\right)^n = A^{-n}$.

Fact 10. (1) *If A is invertible then A^T is also invertible. Moreover,*

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

(2) *If A is invertible and $\alpha \neq 0$, then*

$$(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}.$$

2. HOW TO OBTAIN THE INVERSE OF A MATRIX

There is an algorithm which will allow us to determine if an $n \times n$ matrix A has an inverse, and if it does, will give us A^{-1}:

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ be an $n \times n$ matrix.

(1) Consider the matrix

$$(A \,|\, I_n) = \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & 1 & \cdots & 0 \\ & \vdots & & & \ddots & \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 1 \end{array}\right)$$

(2) Apply Gaussian elimination to the matrix $(A \,|\, I_n)$, so that you obtain $(A \,|\, I_n) \sim (A'' \,|\, B)$.

(3) If $A'' \neq I_n$ then A is not invertible.

(4) If $A'' = I_n$ then A is an invertible matrix whose inverse is the matrix B.

Example 11. We want to determine if $A = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}$ is invertible, and if it is, we want to find A^{-1}.

Proceeding as described above we have:

$$\begin{pmatrix} 0 & 2 & 4 & | & 1 & 0 & 0 \\ 2 & 4 & 2 & | & 0 & 1 & 0 \\ 3 & 3 & 1 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & 1/8 & -5/8 & 3/4 \\ 0 & 1 & 0 & | & -1/4 & 3/4 & -1/2 \\ 0 & 0 & 1 & | & 3/8 & -3/8 & 1/4 \end{pmatrix}$$

Hence

$$A^{-1} = \begin{pmatrix} 1/8 & -5/8 & 3/4 \\ -1/4 & 3/4 & -1/2 \\ 3/8 & -3/8 & 1/4 \end{pmatrix}.$$

Example 12. We want to determine if $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$ is invertible, and if it is, we want to find A^{-1}.

Proceeding as described above we have:

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right).$$

Since A is not equivalent to I_n we have that A has no inverse. In other words, A^{-1} does not exists in this case.

3. Applications to Linear Systems

Theorem 13. *Let A be an $n \times n$ invertible matrix. Then the system $A\vec{x} = \vec{b}$ has a unique solution for any for any $\vec{b}$ in $\mathbb{R}^n$. Moreover, explicitly, the solution is*

$$\vec{x} = A^{-1}\vec{b}.$$

Proof. We can see that $\vec{x} = A^{-1}\vec{b}$ is a solution from

$$A\vec{x} = A\left(A^{-1}\vec{b}\right) = \left(AA^{-1}\right)\vec{b} = I_n\vec{b} = \vec{b}.$$

To see that it is unique assume that $\vec{v}$ and $\vec{v}'$ are two solutions to $A\vec{x} = \vec{b}$. This means that

$$A\vec{v} = \vec{b} = A\vec{v}'.$$

Multiplying the equalities above by A^{-1} from the left, we obtain

$$\vec{v} = A^{-1}\vec{b} = \vec{v}'.$$

Q.E.D.

Example 14. Solve

$$\begin{cases} -2x & +y & = 3 \\ x & +y & = 7 \end{cases}$$

The coefficient matrix associated to the system is

$$A = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Using Theorem 6 we obtain

$$A^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}.$$

Thus,

$$\vec{x} = A^{-1}\,\vec{b} = -\frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} -4 \\ -17 \end{pmatrix}.$$

Theorem 15. *If A is an $n \times n$ matrix then the following are equivalent:*

(1) *A is invertible.*

(2) *The RREF of A is I_n.*

(3) $\operatorname{rank}(A) = n$.

(4) $\operatorname{Col}(A) = \mathbb{R}^n$.

(5) $\operatorname{Row}(A) = \mathbb{R}^n$.

(6) *$A\,\vec{x} = \vec{0}$ has only the trivial solution.*

(7) *$A\,\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^n$.*

(8) *$A\,\vec{x} = \vec{b}$ has exactly one solution for every $\vec{b}$ in $\mathbb{R}^n$.*

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