MATRIX SPACES

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The idea of this lecture is to give a name to the spaces and numbers that have been introduced before. This will give us a better understanding of our basic objects, matrices.

1. Rank of a Matrix

Recall that Gaussian elimination is given in two phases: given a matrix A, by means of elementary row operations we get two matrices

$$A \sim A'' \sim A',$$

where A'' is in REF, and A' is in RREF.

Remark 1. Remember that the leading ones of the REF A' are exactly the same leading ones of the RREF A''. In other words,

$$\begin{aligned} \text{Number of non-zero rows in } A' &= \text{Number of leading ones in } A' \\ &= \text{Number of leading ones in } A'' \\ &= \text{Number of non-zero rows in } A''. \end{aligned}$$

Sometimes the leading ones in either A' or A'' are called **pivots**.

Definition 2. Let A be a matrix. The **rank** of A, denoted as $\operatorname{rank}(A)$, is the number of leading ones in A''.

Example 3. Since

$$A = \begin{pmatrix} 3 & 2 & -2 \\ 1 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we have that $\operatorname{rank}(A) = 3$.

Example 4. Since

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

we have that $\text{rank}(A) = 2$.

The following facts are a direct consequence of the definition of rank.

Fact 5. *Let A be any matrix. Then*

(1) $\text{rank}(A) \leq$ *number of rows in A.*

(2) $\text{rank}(A) \leq$ *number of columns in A.*

(3) *A system of linear equations $A\vec{x} = \vec{b}$ is consistent if and only if* $\text{rank}(A) = \text{rank}(A|\vec{b})$.

Example: *As an example, consider the system*

$$(A|\vec{b}) = \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & k \end{array}\right).$$

In this case $\text{rank}(A) = 2$.

Now, if $k = 0$, we have that $\text{rank}(A|\vec{b}) = 2$ *and the system is consistent.*

If $k \neq 0$, we have that $\text{rank}(A|\vec{b}) = 3$ *and the system is inconsistent.*

(4) *The system $A\vec{x} = \vec{b}$ has infinitely many solutions if and only if*

$$\text{rank}(A|\vec{b}) = \text{rank}(A) < \textit{ the number of columns in } A.$$

(5) *The system $A\vec{x} = \vec{b}$ has unique solution if and only if*

$$\text{rank}(A|\vec{b}) = \text{rank}(A) = \textit{the number of columns in } A.$$

2. Matrix Spaces

Let A be an $m \times n$ matrix.

The columns of A, called the **column vectors** of A, are vectors in $\mathbb{R}^m$. The rows in A, called the **row vectors** of A, are vectors in $\mathbb{R}^n$.

Remark 6. We have that

(1) The system is consistent if and only if $\vec{b}$ is a linear combination of the column vectors of A.
(2) The homogeneous system $A\vec{x} = \vec{0}$ has unique solution ($\vec{x} = \vec{0}$), if and only if the column vectors of A are linearly independent in $\mathbb{R}^m$, if and only if the rank$(A) = n$.

The next is to give a name to the vector spaces related to the vectors mentioned above.

Definition 7. Let A be an $m \times n$ matrix.

The **column space** of A, denoted as $\mathrm{Col}(A)$, is the subspace of $\mathbb{R}^m$ generated (spanned) by the column vectors of A.

The **row space** of A, denoted as $\mathrm{Row}(A)$, is the subspace of $\mathbb{R}^n$ generated (spanned) by the row vectors of A.

Remark 8. (1) Notice that if $A \sim B$ then the rows of B are a linear combination of the rows of A. Now, since $A \sim B$ is equivalent to $B \sim A$, we have that $\mathrm{Row}(A) = \mathrm{Row}(B)$.

(2) Given a matrix A, consider $A' \sim A''$, the REF and RREF matrix associated to A by means of elementary row operations, respectively.

From the first part of this remark we conclude that

$$\mathrm{Row}(A) = \mathrm{Row}(A') = \mathrm{Row}(A'').$$

This is not true for the case of the column space. In other words, $\mathrm{Col}(A) \neq \mathrm{Col}(A')$ and $\mathrm{Col}(A) \neq \mathrm{Col}(A'')$.

Theorem 9. *Let A be a matrix. A basis for* $\mathrm{Row}(A)$ *is given by the non-zero rows of an equivalent matrix of A in REF.*

Example 10. From

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \\ 2 & 5 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

we conclude:

- $\mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$
  $= \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$
- $\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 4 \\ 7 \end{pmatrix} \right\}.$
- $\mathrm{rank}(A) = 3.$

Also notice that $3 = \mathrm{rank}(A) = \dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A))$.

Example 11. From

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & -1 & -1 \\ 1 & 5 & -2 & 4 \\ 1 & -1 & 0 & -2 \\ 1 & 1 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we conclude:

- $$\mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -3 \end{pmatrix} \right\}$$
  $$= \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$
- $$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \right\}.$$
- $\mathrm{rank}(A) = 3$.

Also notice that

$$3 = \mathrm{rank}(A) = \dim\left(\mathrm{Col}(A)\right) = \dim\left(\mathrm{Row}(A)\right).$$

3. Rank Theorem

Theorem 12 (Rank Theorem)**.** *Let A be an $m \times n$ matrix. Then*

$$\mathrm{rank}(A) = \dim\left(\mathrm{Col}(A)\right) = \dim\left(\mathrm{Row}(A)\right).$$

Proof. The theorem is just a consequence of the fact that these three numbers are equal to the number of leading ones in the RREF of A. Q.E.D.

4. Nullspace of a Matrix

Definition 13. Let A be an $m \times n$ matrix.

The **null space** of A, denoted as $\text{Null}(A)$, is the subspace of $\mathbb{R}^n$ given by all the solutions of the homogeneous linear system $A\vec{x} = \vec{0}$.

Theorem 14. *Let A be an $m \times n$ matrix. Then any vector in* $\text{Row}(A)$ *is orthogonal to any vector in* $\text{Null}(A)$*.*

Proof. Let $\vec{v}_i$ be the $i - th$ row vector of A. Let $\vec{u}$ be a vector in $\text{Null}(A)$, so by definition of the nullspace we have that

$$A\vec{u} = \vec{0}. \tag{1}$$

But by definition of matrices, equation (1) implies that $\vec{v}_i \bullet \vec{u} = 0$, for any i. Thus, $\vec{u}$ will be orthogonal to any linear combination of the row vectors $\vec{v}_i$'s. Q.E.D.

Theorem 15. *Let A be an $m \times n$ matrix. Let $\{\vec{w}_1, \ldots, \vec{w}_n\}$ be a basis for $\mathbb{R}^n$. Then* $\text{Col}(A) = \text{Span}\{A\vec{w}_1, \ldots, A\vec{w}_n\}$*.*

5. Dimension Theorem

Next, one of the most important results in Linear Algebra.

Theorem 16 (Dimension Theorem)**.** *Let A be an $m \times n$ matrix. Then*

$$\text{rank}(A) + \dim\left(\text{Null}(A)\right) = n.$$

Some consequences of the dimension theorem and its proof are the following.

(1) Let A be an $m \times n$ matrix. Assume that

$$\text{Span}\{\vec{u}_1, \ldots, \vec{u}_d\}$$

is the solution set for $A\vec{x} = \vec{0}$ obtained using the method that we saw in the second lecture on linear systems.

Then $\{\vec{u}_1, \ldots, \vec{u}_d\}$ is a basis for $\text{Null}(A)$.

(2) Given an $m \times n$ matrix, a basis for $\mathbb{R}^n$ is given by putting together the vectors in a basis for $\text{Null}(A)$ and the vectors in a basis for $\text{Row}(A)$.

(3) If $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is a basis for $\text{Row}(A)$, then $\{A\vec{u}_1, \ldots, A\vec{u}_d\}$ is a basis for $\text{Col}(A)$.

Example 17. From

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \\ 3 & 5 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we have that

- Rank Theorem: $\text{rank}(A) = \text{Col}(A) = \text{Row}(A) = 3$.
- Dimension Theorem:

$$\begin{aligned} n &= \text{rank}(A) + \dim(\text{Null}(A)) \\ 3 &= \quad 3 \quad + \quad 0 \end{aligned}$$

  so $\text{Null}(A) = \{\vec{0}\}$.
- A basis for $\text{Col}(A)$ is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right\}$.
- A basis for $\text{Row}(A)$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.
- A basis for the nullspace of A is the empty space $\{\emptyset\}$, since $\text{Null}(A) = \{\vec{0}\}$.

Example 18. From

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

we have that

- Rank Theorem: $\text{rank}(A) = \text{Col}(A) = \text{Row}(A) = 2$.
- Dimension Theorem:

  $$\begin{aligned} n &= \text{rank}(A) + \dim(\text{Null}(A)) \\ 3 &= \quad 2 \quad + \quad 1 \end{aligned}$$

  so $\text{Null}(A) \neq \{\vec{0}\}$.
- A basis for $\text{Col}(A)$ is $\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$.
- A basis for $\text{Row}(A)$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$.
- To obtain a basis for $\text{Null}(A)$ we need to proceed as before: from

  $$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

  we have that

  $$x_1 = x_3, \qquad x_2 = -2x_3.$$

  Set $x_3 = t$, then the solution set for the system $A\vec{x} = \vec{0}$ is

  $$\left\{ \begin{pmatrix} t \\ -2t \\ t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

  so a basis for $\text{Null}(A)$ is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$.

Example 19. From

$$\begin{pmatrix}1 & 2 & -1 & 3\\ 2 & 4 & -1 & 6\\ 0 & 1 & 0 & 2\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 0\end{pmatrix}$$

we have that

- Rank Theorem: $\text{rank}(A) = \text{Col}(A) = \text{Row}(A) = 3$.
- Dimension Theorem:

$$\begin{aligned} n &= \text{rank}(A) + \dim(\text{Null}(A)) \\ 4 &= \quad 3 \quad + \quad 1 \end{aligned}$$

  so $\text{Null}(A) \neq \{\vec{0}\}$.
- A basis for $\text{Col}(A)$ is $\left\{ \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}2\\4\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\end{pmatrix} \right\}$.
- A basis for $\text{Row}(A)$ is $\left\{ \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\2\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\}$.
- To obtain a basis for $\text{Null}(A)$ we need to proceed as before: from

$$\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 0\end{pmatrix}$$

  we have that $x_1 = x_4$, $x_2 = -2x_4$, and $x_3 = 0$. Set $x_4 = t$, then the solution set for the system $A\vec{x} = \vec{0}$ is

$$\left\{ \begin{pmatrix}t\\-2t\\0\\t\end{pmatrix} \mid t \in \mathbb{R} \right\}$$

  so a basis for $\text{Null}(A)$ is $\left\{ \begin{pmatrix}1\\-2\\0\\1\end{pmatrix} \right\}$.

6. Applications on Linear Systems

The following theorem is a direct interpretation of the method of solving linear systems by Gaussian elimination.

Theorem 20. *Let A be an $n \times n$ matrix. Then the following are equivalent:*

(1) *The RREF of A is I_n.*
(2) $A\vec{x} = \vec{0}$ *has only the trivial solution.*
(3) $A\vec{x} = \vec{b}$ *has exactly one solution for any $\vec{b}$ in $\mathbb{R}^n$.*
(4) $A\vec{x} = \vec{b}$ *is consistent for any $\vec{b}$ in $\mathbb{R}^n$.*

Some consequences of the Rank Theorem are the following.

Theorem 21. *Let A be an $m \times n$ matrix. Then the following are equivalent:*

(1) *The homogeneous linear system $A\vec{x} = \vec{0}$ has only the trivial solution.*
(2) *The column vectors of A are linearly independent.*
(3) $\text{rank}(A) = n$.

Theorem 22. *Let A be an $m \times n$ matrix. Then the following are equivalent:*

(1) *The linear system $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$ in $\mathbb{R}^m$.*
(2) $\text{Col}(A) = \mathbb{R}^m$.
(3) $\text{rank}(A) = m$.

Some corollaries of these results and fact 5 are:

Corolary 23. *Let A be an $m \times n$ matrix. Then*

(1) *If $m > n$, then the system $A\vec{x} = \vec{b}$ is not consistent for any $\vec{b}$ in $\mathbb{R}^m$.*
(2) *If $m < n$, then the system $A\vec{x} = \vec{b}$ is either inconsistent, or it has infinitely many solutions.*

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