MATRIX ALGEBRA

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Matrices are objects that play various roles in the study of vector spaces:

- Matrices form one of the most basic examples of vector spaces.
- The problem of solving linear systems can be stated and solved in terms of matrices.
- (Later we will see that) matrices are the "good" functions between vector spaces.

A characteristic of matrices that makes us try to express problems in linear algebra in terms of them is that matrices behave like numbers. The goal of this lecture is to establish the basic algebraic properties of matrices.

1. Matrix Algebra

1.1. **Basic Arithmetic of Matrices.** Let $\mathcal{M}_{m\times n}$ denote the vector space of all the $m \times n$ matrices. Let A and B two $m \times n$ matrices.

Recall that we say that A and B are **equal** if they are equal entry-wise. In other words, if $A = (a_{ij})$ and $B = (b_{ij})$, then $A = B$ if and only if $a_{ij} = b_{ij}$ for all i and j.

We defined the **addition** $A + B$ entry-wise: $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$. Similarly, we defined the **difference** $A - B$ entry-wise.

An important matrix in $\mathcal{M}_{m\times n}$ is the **zero matrix**, denoted as $\boldsymbol{0}$.

We also defined **scalar product** as $\alpha(a_{ij}) = (\alpha a_{ij})$. Moreover, we defined the **negative** of a matrix A as $-A := (-1)A$.

1.2. **Matrix Multiplication.** We need to fix some notation first. If A is the matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

the $i-th$ row of A will be denoted by $\mathbf{A}_i$, namely

$$\mathbf{A}_i = \begin{pmatrix} a_{i1} & \cdots & a_{in} \end{pmatrix},$$

while the $j-th$ column of A will be denoted by $\mathbf{A}^j$, namely

$$\mathbf{A}^j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

Definition 1. Let A be an $m \times n$ matrix. Let B be a $n \times p$ matrix. The **product** of A and B is the $m \times p$ matrix whose (i, j)-entry is the dot product of the $i-th$ row of A with the $j-th$ column of B. More precisely,

$$AB = \begin{pmatrix} \mathbf{A}_1 \bullet \mathbf{B}^1 & \mathbf{A}_1 \bullet \mathbf{B}^2 & \cdots & \mathbf{A}_1 \bullet \mathbf{B}^p \\ \mathbf{A}_2 \bullet \mathbf{B}^1 & \mathbf{A}_2 \bullet \mathbf{B}^2 & \cdots & \mathbf{A}_2 \bullet \mathbf{B}^p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_m \bullet \mathbf{B}^1 & \mathbf{A}_m \bullet \mathbf{B}^2 & \cdots & \mathbf{A}_m \bullet \mathbf{B}^p \end{pmatrix}$$

Example 2. Some examples of product of matrices are:

$$\begin{pmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -1 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 15 & 0 \\ 4 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 \\ -1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ -3 & -5 \\ 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix} \left[\begin{pmatrix} 3 & 4 \\ -1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix} \right] = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} -1 & 5 \\ -3 & -5 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 15 & 45 \\ -2 & 6 \end{pmatrix}$$

Remark 3. (1) Notice that AB can be defined only when the number of columns of A is equal to the number of rows of B:

$$(m \times n)(n \times p) = m \times p$$

(2) The product of matrices is a generalization of dot product.

(3) The product of matrices IS NOT given by a product entry-wise.

(4) The product of matrices IS NOT commutative in general, even when both products AB and BA are defined.

In other words, in general $AB \neq BA$.

As an example consider the products

$$\begin{pmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 15 & 5 \\ -3 & -5 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} -1 & 45 \\ -2 & 6 \end{pmatrix}$$

and

$$\begin{pmatrix} 15 & 5 \\ -3 & -5 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 35 & 30 & 85 \\ -11 & -18 & -25 \\ 16 & 28 & 36 \end{pmatrix}.$$

(5) We can have that $AB = \mathbf{0}$, but neither A or B are the zero matrices.

As an example consider the product of matrices

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ -2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(6) The cancellation law DOES NOT hold. In other words, we might have that $AB = AC$, but $B \neq C$.

Similarly, we might have that $BA = CA$, but $B \neq C$.

As an example consider the products of matrices

$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}.$$

But notice that

$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \end{pmatrix}.$$

Remember that for any n we have the $n \times n$ **identity matrix** I_n, which is the matrix whose entries along the diagonal are 1, and zero elsewhere:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If the size of I_n is clear, then we will denote the $n \times n$ identity matrix as I.

The main properties of product of matrices are summarized in the following.

Theorem 4. *Let A, B and C matrices in $\mathcal{M}_{m\times n}$ for which the following operations are defined. Then*

(1) $A + B = B + A$.

(2) $A + (B + C) = (A + B) + C$.

(3) $A - A = \mathbf{0}$.

(4) $A + \mathbf{0} = \mathbf{0} + A = A$.

(5) $1A = A$.

(6) $0A = \mathbf{0}$.

(7) $\alpha(A + B) = \alpha A + \alpha B$.

(8) $(\alpha + \beta)A = \alpha A + \beta A$.

(9) $A(BC) = (AB)C$.

(10) $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

(11) $A(B + C) = AB + AC$.

(12) $(A + B)C = AC + BC$.

(13) $(AB)^T = B^T A^T$.

(14) $\mathbf{0}A = \mathbf{0}$, *and* $\mathbf{0}B = \mathbf{0}$.

(15) $IA = A$, *and* $BI = B$.

1.3. **Product in $\mathbb{M}_{n\times n}$.** Recall that a square matrix is an $n\times n$ matrix. The product of matrices of square matrices has an extra property: we can always consider the "powers" of them.

More precisely, given a $n \times n$ matrix A, we defined

$$A^k = A \cdot A \cdots A \qquad n \text{ factors.}$$

We have the convention that $A^0 = I_n$.

Remark 5. (1) We still might have that $A^k = \mathit{0}$, but $A \neq \mathit{0}$.
As an example consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that $A^3 = \mathit{0}$.

(2) We have that "FOIL" does not hold in general:

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B) = A(A+B) + B(A+B) \\ &= A^2 + AB + BA + B^2.\end{aligned}$$

So, unless the product of A and B is commutative, FOIL will not hold.

(3) We can evaluate polynomials on square matrices: if

$$P(x) = a_r x^r + \cdots + a_1 x + a_0 x^0$$

is a polynomial, then we can evaluate $P(x)$ on an $n \times n$ matrix A as

$$P(A) = a_r A^r + \cdots + a_1 A + a_0 I_n.$$

As an example, we consider the polynomial

$$P(x) = 3x^2 + x - 2.$$

Evaluating at

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

we obtain

$$P(A) = 3\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^2 + \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 26 \\ 0 & 28 \end{pmatrix}.$$

2. Back to Linear Systems

Using the product of matrices we can write a system of linear equations so that we can apply the algebraic properties of matrices. Given a system of linear equations

$$\text{(S)} \qquad \begin{cases} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = b_1 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = b_2 \\ & & \vdots & & \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = b_m \end{cases}$$

we can consider the augmented matrix $(A|\vec{b})$, where $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ is the coefficient matrix of (S), and where $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ is the coefficient vector of (S). Thus, the system (S) can be written as

$$(A|\vec{b}) = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right).$$

Now set $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Since

$$A\vec{x} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n \\ & & \vdots & \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n \end{pmatrix}$$

we recover the system (S) as $\quad A\vec{x} = \vec{b}$:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

Summarizing, a system (S) can be written either as $(A|\vec{b})$, or as $A\vec{x} = \vec{b}$.

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