SYSTEMS OF LINEAR EQUATIONS - PART 2

JOSÉ MALAGÓN-LÓPEZ

We saw that if the coefficient matrix associated with a system of linear equations (S) is in RREF, then it is not difficult to obtain the solution set of (S). In this lecture we will complete the task of finding the solution set for a given system of linear equations. We will develop a method that will produce a new system with the same solutions as the original system, but whose coefficient matrix is in RREF. Such method is called Gaussian elimination.

1. Elementary Row Operations

The idea is to consider operations for rows in a matrix such that after performing them we can eliminate entries. If the matrix under consideration is the augmented matrix for a system of linear equations, then the elimination of entries will mean that we are eliminating variables in the equations.

Let A be an $m \times n$ matrix. The following operations on the rows of A are called the **elementary row operations**:

(1) Interchange any two rows of A.

(2) Multiply any row of A by a non-zero number.

(3) Add the product of any row A by a number to another row of A.

Remark. If $(A|\mathbf{b})$ is the augmented matrix associated to a system of equations (S), then the operations described above correspond to:

(1) Interchanging any two equations in (S).

(2) Multipling any equation in (S) by a non-zero number.

(3) Adding the product of any equation in (S) by a number to another equation in (S).

Since it is more convenient to work exclusively with the coefficients, we will describe the method in terms of matrices.

Remark. We could have defined the same operations for columns, obtaining similar results. We will restrict ourselves to the case of row operations only.

Let A and B be two matrices. We say that A and B are **equivalent**, and denoted by $A \sim B$, if we can transform one of the matrices into the other after performing a finite sequence of elementary row operations.

Example. We have that

$$\begin{pmatrix} 2 & 1 & 6 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 7/3 \\ 0 & 1 & 4/3 \end{pmatrix}$$

Indeed, starting with $\begin{pmatrix} 2 & 1 & 6 \\ 1 & -1 & 1 \end{pmatrix}$, interchange the two rows:

$$\begin{pmatrix} 2 & 1 & 6 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 6 \end{pmatrix}$$

Add -2 times the first row to the second row:

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 4 \end{pmatrix}$$

Multiply by 1/3 the second row:

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 4/3 \end{pmatrix}$$

Notice that the matrix obtained is in REF. Finally, add the second row to the first row:

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 4/3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 7/3 \\ 0 & 1 & 4/3 \end{pmatrix}$$

We have obtained a matrix in RREF.

2. Equivalent Systems

Two systems of linear equations are called **equivalent** if they have the same solution set.

The importance of similar matrices lies in the following theorem.

Theorem. Let (S) and (S') be two systems of linear equations. Suppose that the systems can be expressed in terms of augmented matrices as $(A|\mathbf{b})$, and $(A'|\mathbf{b}')$, respectively. If $(A|\mathbf{b}) \sim (A'|\mathbf{b}')$ then the systems are equivalent.

Example. We have that the vector $\begin{pmatrix} 7/3 \\ 4/3 \end{pmatrix}$ is the solution set for the system

$$\left(\begin{array}{cc|c} 1 & 0 & 7/3 \\ 0 & 1 & 4/3 \end{array}\right).$$

Also, in the previous example we saw that $\left(\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 7/3 \\ 0 & 1 & 4/3 \end{array}\right)$.

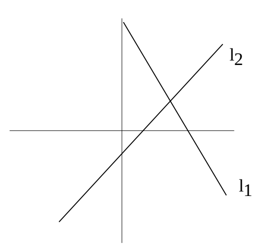
From the theorem above we have that the vector $\begin{pmatrix} 7/3 \\ 4/3 \end{pmatrix}$ is the solution set for the system associated with the augmented matrix

$$\left(\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & -1 & 1 \end{array}\right).$$

Remark. Lets use the previous example to see what is going on from a geometric point of view. Recall that any linear equation in two variables defines a line in $\mathbb{R}^2$. In this case, the lines

$$l_1 : 2x + y = 6, \qquad \text{and} \qquad l_2 : x - y = 1$$

are represented in the following graph:



Also recall that the solution set of the system of linear equations

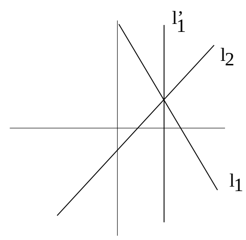
$$\text{(S)} \qquad \begin{cases} x & -y & = & 1 \\ 2x & +y & = & 6 \end{cases}$$

consists of the intersection of ℓ_1 and ℓ_2.

In the first step we fixed the line $x - y = 1$ and substituted the line $2x + y = 6$ in the system by the line $\ell_1' : \; 3y = 4$, which also can be described as $y = 4/3$.

$$\begin{cases} x & -y & = & 1 \\ 2x & +y & = & 6 \end{cases} \sim \begin{cases} x & -y & = & 1 \\ & y & = & 4/3 \end{cases}$$

This is the key point in the process: ℓ_1 and ℓ_1' are different, but the intersection of ℓ_1 with ℓ_2 is the same that the intersection of ℓ_1' with ℓ_2.

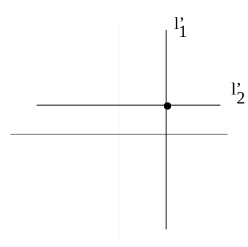


The second step consists of fixing the line ℓ_1', which is parallel to a coordinate axis, and substituting ℓ_2 by a new line, say ℓ_2', which will be perpendicular to ℓ_1'. The new line ℓ_2' will be obtained by rotating ℓ_2, taking as pivot the intersection set in consideration: we exchange $x - y = 1$ by $x = 7/3$. Thus we have equivalent systems

$$\begin{cases} x & -y & = & 1 \\ 2x & +y & = & 6 \end{cases} \sim \begin{cases} x & -y & = & 1 \\ & y & = & 4/3 \end{cases} \sim \begin{cases} x & & = & 7/3 \\ & y & = & 4/3 \end{cases}$$

whose solution set consists of the intersection of the lines

$$\ell_2' : x = 7/3 \qquad \text{and} \qquad \ell_1' : y = 4/3$$



3. Gaussian Elimination

The process of transforming a matrix into a matrix in RREF by means of elementary row operations is called Gaussian elimination. It consists of two phases: a forward process (the first 6 steps), and a backwards process (the last 2 steps). The forward process will transform the matrix into a matrix in REF, while the backwards process will play the role of back substitution.

Gaussian Elimination. Let A be a matrix of m by n. By means of the elementary row operations proceed as follows:

(G1) In the left-most non-zero column, obtain a 1 and call it a leading one.

(G2) Place the row containing a leading one at the top of the matrix.

(G3) Using the row containing a leading one, obtain zeros in the entries below a leading one.

(G4) Obtain a 1 in the next left-most non-zero possible column in a row not containing a leading one.

(G5) Place the row containing the new leading one immediately below the row containing the previous leading one.

(G6) Repeat steps (G3)-(G5) for the new leading one, until there is no more non-zero rows.

(G7) Using the last non-zero row, obtain zeros above the leading one in such row.

(G8) Repeat step (G7) for each preceding row.

Theorem. Gaussian elimination transforms any matrix into a UNIQUE matrix in RREF.

In other words, the outcome of Gaussian elimination does not depend on the choices made in the process. The matrix obtained by applying Gaussian elimination to a matrix A is called **the reduced row echelon form of** A.

Example of Gaussian Elimination.

We start with the same matrix (the matrix in the middle) and we take two different sequences of elementary row operations:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 3 & 2 & -2 & 1 \\ 1 & 2 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 2 & -2 & 1 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 1 & 0 & 3 \\ 3 & 2 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & -2 & -8 \\ 0 & 1 & -1 & -1 \end{array}\right) \sim \qquad\qquad \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & -4 & 1 & -5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 8 \\ 0 & 1 & -1 & -1 \end{array}\right) \sim \qquad\qquad \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -4 & 1 & -5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & -3 & -9 \end{array}\right) \sim \qquad\qquad \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -3 & -9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right) \sim \qquad\qquad \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \sim \qquad\qquad \sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \sim \qquad\qquad \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

4. Solving a System of Linear Equations

Now we are in a position of describing the complete algorithm that will determine if a system of linear equations is consistent or not, and in the case of being consistent, it will give us the system's solution set by finding a basis for the homogeneous system associated to the system and a particular solution of the system, all of this SIMULTANEOUSLY!

Solving a System of Linear Equations. Let (S) be a system of m linear equations in n unknowns

$$\begin{cases} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = b_1 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = b_2 \\ & & \vdots & & \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = b_m \end{cases}$$

Let $(A \mid \mathbf{b})$ be the augmented matrix of the system. Then:

(A1) Apply Gaussian elimination to $(A \mid \mathbf{b})$.

Let $(A' \mid \mathbf{b}')$ be the reduced row echelon form of $(A \mid \mathbf{b})$.

(A2) Determine if the system is consistent.
Recall that the system will be inconsistent if $(A' \mid \mathbf{b}')$ contains a row of the form $\left(\begin{array}{ccccc|c} 0 & 0 & \cdots & 0 & 0 & b \end{array}\right)$, with $b \neq 0$.

If the system is consistent then:

(A3) Solve for the left-most variables in each equation.

(A4) Assign a parameter to each free variable.

(A5) Express the solution set in terms of these parameters.

Thus, if $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is a basis for the homogeneous system associated to the system (S), and $\mathbf{s}$ is a solution of the system (S), then any solution of the system is of the form

$$\mathbf{s} + \alpha_1 \mathbf{v}_1 + \cdots + \alpha_m \mathbf{v}_m,$$

for some scalars $\alpha_1, \ldots, \alpha_m$.

Example. Solve the following system of linear equations:

$$\begin{cases} x & +y & +2z & = & 9 \\ 2x & +4y & -3z & = & 1 \\ 3x & +6y & -5z & = & 0 \end{cases}$$

We have that

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Solving for the left-most variables we get

$$x_1 = 1, \qquad x_2 = 2, \qquad x_3 = 3.$$

Thus, the solution set consists of the vector

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

Example. Solve the following system of linear equations:

$$\begin{cases} x_1 & +2x_2 & -x_3 & = & -1 \\ 2x_1 & +2x_2 & +x_3 & = & 1 \\ 3x_1 & +5x_2 & -2x_3 & = & -1 \end{cases}$$

We have that

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Solving for the left-most variables we get

$$x_1 = 4, \qquad x_2 = -3, \qquad x_3 = -1.$$

Thus, the solution set consists of the vector

$$\left\{ \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} \right\}$$

Example. Solve the following system of linear equations:

$$\begin{cases} x_1 & +2x_2 & -x_3 & +3x_4 & = & 2 \\ 2x_1 & +4x_2 & -x_3 & +6x_4 & = & 5 \\ & x_2 & & +2x_4 & = & 3 \end{cases}$$

We have that

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

Solving for the left-most variables we get

$$x_1 = -3 + x_4, \qquad x_2 = 3 - 2x_4, \qquad x_3 = 1.$$

Set $x_4 = t$. Thus, the solution set consists of the vector

$$\left\{ \begin{pmatrix} -3+t \\ 3-2t \\ 1 \\ t \end{pmatrix} \mid t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} -3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

In this case, the vector

$$\begin{pmatrix} -3 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

is a particular solution of the system, while the line

$$t \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

is the solution set of the associated homogeneous system.

Example. Determine for which values of k does the system below have exactly one/infinitely many/no solution.

$$\begin{cases} x & -y & = & 3 \\ 2x & -2y & = & k \end{cases}$$

Applying Gaussian elimination we obtain

$$\left(\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & -2 & k \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & k-6 \end{array}\right)$$

Now notice that

- The system has no solution if and only if the RREF of the augmented matrix looks like
$$\left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & k-6 \neq 0 \end{array}\right)$$
Thus, for $k \neq 6$ the system has no solution.
- If $k = 6$, then the augmented matrix takes the form
$$\left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & k-6 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$
Such system is consistent, and its solution set is given as follows:
$$x_1 = 3 + x_2.$$
Set $x_2 = t$, so the solution set consists of the vectors
$$\begin{pmatrix} 3+t \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Conclusion, for $k = 6$ the system has infinitely many solutions, while for $k = 6$ the system has no solution. Also, there is no value of k for which the system will have a unique solution.

E-mail address: `jmalagon@uottawa.ca`