SYSTEMS OF LINEAR EQUATIONS - PART 1

JOSÉ MALAGÓN-LÓPEZ

A recurrent situation so far has been to determine the solution set of a system of linear equations. In this lecture we start the study of systems of linear equations so that we can improve our techniques used so far. A first step will be to express the solution set in terms of vector spaces.

1. Systems of Equations

1.1. **Basic Definitions.** A linear equation in n unknowns is an algebraic expression of the form

$$(*) \qquad a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $x_1, \ldots, x_n$ are unknowns (variables), $a_1, \ldots, a_n$ are real numbers called the **coefficients of the equation**, and b is a real number called the **constant term of the equation**. A solution to the equation (*) is a vector

$$\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \in \mathbb{R}^n$$

such that the equation holds when the x_i's are substituted by the numbers s_i's:

$$a_1(s_1) + a_2(s_2) + \cdots + a_n(s_n) = b.$$

Example. A solution to the equation

$$2x_1 + 3x_2 - x_3 = -5$$

is the vector

$$\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$$

since

$$2(-2) + 3(0) - (1) = -5.$$

A **system of m linear equations in n unknowns** is a collection

$$(\mathrm{S}) \qquad \begin{cases} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = b_1 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = b_2 \\ & & \vdots & & \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = b_m \end{cases}$$

consisting of m linear equations in n unknowns. The real numbers a_{ij}, $1 \le i \le m$, $1 \le j \le n$, are called the **coefficients of the system** (S). The real numbers $b_1, \ldots, b_m$ are called the **constant terms of the system** (S).

A solution to the system (S) is a vector

$$\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \in \mathbb{R}^n$$

that is a solution for EACH of the linear equations in the system (S).

The collection of all the solutions to a system (S) is called the **solution set** of (S). If the solution set has at least one solution we say that the system is **consistent**. If the solution set has no solution (in other words, it is empty) we say that the system is **inconsistent**.

Example. The vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ is a solution for the system

$$\begin{cases} -x_1 & +4x_2 & = & 14 \\ 2x_1 & -x_2 & = & -7 \end{cases}$$

Example. The vector $\begin{pmatrix} 9 \\ -13 \\ -1 \end{pmatrix}$ is a solution for the system

$$\begin{cases} 5x_1 & +3x_2 & +7x_3 & = & -1 \\ 2x_1 & & -x_3 & = & 19 \end{cases}$$

Example. Unlike the systems in the previous two examples, the following system is inconsistent.

$$\begin{cases} x_1 & +x_2 & = & 1 \\ x_2 & +x_2 & = & 2 \end{cases}$$

Example. The vectors $\begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -6 \\ 1 \end{pmatrix}$ are solutions for the system

$$\begin{cases} x_1 & -x_2 & +2x_3 & = & 1 \\ -x_1 & & +x_3 & = & 2 \end{cases}$$

1.1.1. *Homogeneous Systems.* A linear system is called **homogeneous** if all the constant terms are equal to zero:

$$\begin{cases} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = 0 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = 0 \\ & & \vdots & & \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = 0 \end{cases}$$

Remark. Every homogeneous system has at least one solution, the trivial solution

$$\boldsymbol{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n.$$

1.2. **Solution Set.** Given a system of linear equations

$$\text{(S)} \qquad \begin{cases} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = b_1 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = b_2 \\ & & \vdots & & \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = b_m \end{cases}$$

the **homogeneous system corresponding** to (S) is the system

$$\begin{cases} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = 0 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = 0 \\ & & \vdots & & \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = 0 \end{cases}$$

Theorem.

(1) The solution set of a homogeneous system of m linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

(2) Let s be a solution of a system (S) of m linear equations in n unknowns. Let K_H be the solution set of the homogeneous system corresponding to (S). Then the solution set of the system (S) is the set of vectors of the form

$$\{s + v \mid v \in K_H\}.$$

Remark. In particular we have that the solution set of a system of linear equations can be either: empty, a unique vector, or an infinite number of vectors.

Remark. A more explicit description of the second part of the theorem above is the following:

Consider a system (S) of m linear equations in n unknowns. Assume that the system has at least one solution. Let $s \in \mathbb{R}^n$ be such a solution. Then

- If the solution set of its homogeneous system consists only of the zero vector in $\mathbb{R}^n$, then s is the unique solution of (S).
- If the solution set of its homogeneous system is the subspace in $\mathbb{R}^n$ with basis $\{v_1, \ldots, v_l\}$, then any solution for the system (S) is of the form

$$s + \alpha_1 v_1 + \cdots + \alpha_l v_l,$$

where $\alpha_1, \ldots, \alpha_l$ are real numbers.

Remark. Notice that a non-homogeneous system might have no solution, but its associated homogeneous system might have an infinite number of solutions.

2. Solving Systems of Equations

We will develop techniques for determining if a system of equations is consistent, at the same time obtaining the solution set if it exists.

First, we consider an easy case.

Example 1. Find the solution set of $\begin{cases} x_1 & +2x_2 & +x_3 & -x_4 & = & 2 \\ & x_2 & & -x_4 & = & -1 \\ & & & x_4 & = & 3 \end{cases}$

We proceed in an organized way:

- Solve for the left-most variables:

$$\begin{aligned} x_1 &= 2 - 2x_2 - x_3 + x_4 \\ x_2 &= -1 + x_4 \\ x_4 &= 3 \end{aligned}$$

- Apply back substitution and obtain

$$\begin{aligned} x_1 &= 1 - x_3 \\ x_2 &= 2 \\ x_4 &= 3 \end{aligned}$$

  Thus, the solution set of the system consists of all the vectors of the form

$$\begin{pmatrix} 1 - x_3 \\ 2 \\ x_3 \\ 3 \end{pmatrix}$$

- We have no conditions on x_3, so it can be considered as a parameter in the description of the solution set, say

$$x_3 = t.$$

Thus, the solution set of the system in parametric form is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - t \\ 2 \\ t \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Now we consider a simpler case, which will be a "model situation".

Example 2. Find all the solutions of the system

$$\begin{cases} x_1 + 3x_2 + 2x_5 = 0 \\ x_3 + 2x_4 = 1 \\ x_6 = 3 \end{cases}$$

We proceed as before:

- Solve for the left-most variables:

$$x_1 = -3x_2 - 2x_5$$
$$x_3 = 1 - 2x_4$$
$$x_6 = 3$$

- We have no condition s on x_2, x_4 and x_5, so they will be parameters in the description of the solution set, say

$$x_2 = r, \quad x_4 = s, \quad x_5 = t.$$

  Thus, the solution set for the system consists of the vectors whose components are of the form

$$x_1 = -3r - 2t,\ x_2 = r,\ x_3 = 1 - 2s,\ x_4 = s,\ x_5 = t,\ x_6 = 3.$$

We conclude that the solution set of the system in parametric form is

$$\begin{pmatrix} -3r - 2t \\ r \\ 1 - 2s \\ s \\ t \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 3 \end{pmatrix} + r \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Notice that in this example we do not need to apply back substitution.

Also notice that by the theorem of the previous section we have that the solution set of the associated homogeneous system is the subspace

$$\text{Span}\left\{\begin{pmatrix}-3\\1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\-2\\1\\0\\0\end{pmatrix},\begin{pmatrix}-2\\0\\0\\0\\1\\0\end{pmatrix}\right\}.$$

3. Augmented Matrices

In order to give an explicit algorithm for obtaining the solution set of a system of linear equations we need some new terminology.

Consider a system of linear equations

$$\text{(S)}\qquad \begin{cases} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = b_1\\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = b_2\\ & & \vdots & & \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = b_m \end{cases}$$

The matrix $\begin{pmatrix} a_{11} & \cdots & a_{1n}\\ & \ddots & \\ a_{m1} & \cdots & a_{mn}\end{pmatrix}$ is called the **coefficient matrix** of (S).

The vector $\begin{pmatrix} b_1\\ \vdots\\ b_m\end{pmatrix}$ is called the **constant vector** of (S).

The matrix $\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1\\ & \ddots & & \vdots\\ a_{m1} & \cdots & a_{mn} & b_m\end{array}\right)$ is called the **augmented matrix** of (S).

If A denotes the coefficient matrix of a system (S), and $\boldsymbol{b}$ denotes the constant term vector of the system, sometimes we will denote the system (S) as the augmented matrix

$$(A|\boldsymbol{b}).$$

Note. If the coefficient matrix of a system is in RREF, and the augmented matrix has a row of the form

$$\left(\begin{array}{ccccc|c} 0 & 0 & \cdots & 0 & 0 & b \end{array}\right),$$

with $b \neq 0$, then the system is inconsistent.

4. Reduced Row Echelon Forms

The coefficient matrix of the system used in Example 3 is an example of the following type of matrices.

A matrix is said to be in **row-echelon form (REF)** if it has the following properties:

(1) The first non-zero entry (if any) in each row is 1, called the leading one.
(2) All the rows consisting entirely of zeros are located at the bottom of the matrix.
(3) The leading one of a row is located in a column to the right with respect to the leading ones of the preceding rows.

A matrix in row-echelon form is said to be in **reduced row-echelon form (RREF)** if the following additional condition is satisfied

(4) The leading one in each row is the only non-zero entry in its column.

Remark. Notice that (1), (2) and (3) are requirements for the rows, while (4) is a column requirement.

Examples of RREF.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Examples of REF.

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrices that are not REF.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

5. Obtaining the Solution Set for a System Whose Augmented Matrix is in RREF

If the coefficient matrix of a system of linear equations is in RRE form, we can find the solution set of the system in parametric form as follows:

S1. Solve for the left-most variables in each equation. The variables on the right side of the equation will be called **free**.

S2. Assign a parameter to each free variable.

S3. Express the solution set in terms of these parameters.

Example. Find the solution set for the system whose augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & -1 & 5 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{array}\right)$$

Notice that the coefficient matrix is in RREF, and that the (actual) system is

$$\begin{cases} x_1 - x_2 - x_5 = 5 \\ x_3 + x_5 = -2 \\ x_4 = 2 \end{cases}$$

Following the algorithm outlined above we get:

S1 Solve for the left-most variables:

$$x_1 = 5 + x_2 + x_5$$
$$x_3 = -2 - x_5$$
$$x_4 = 2$$

S2 The variable x_2 and x_5 are free. Set $x_2 = s$, and $x_5 = t$.

S3 We conclude that the solution set of the system in parametric form is

$$\begin{pmatrix} 5+s+t \\ s \\ -2-t \\ 2 \\ t \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -2 \\ 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Remark. Assume that the augmented matrix of a consistent system is in RREF. Then

- If the system is homogeneous and it has a non trivial solution, then the vectors parameterizing the solution set obtained by this method is a basis for the solution set.
- Let n be the number of variables in the system. Let m be the number of non-zero rows in the augmented associated matrix. Then the number of parameters in the solution set is equal to $n - m$.

E-mail address: `jmalagon@uottawa.ca`