

BASES AND DIMENSION

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Given a vector space $V = \text{Span}(S)$, the generator set S might not be good enough for working with V . One problem is that a vector v in V might be written in more than one way as a linear combination of the vectors in S .

For example, consider the subspace $\text{Span}(S)$ of \mathbb{R}^3 , where

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} \right\}.$$

The vector $\begin{pmatrix} 1 \\ 9 \\ 2 \end{pmatrix}$ is in $\text{Span}(S)$ and can be expressed as a linear combination of the vectors in S in different ways:

$$\begin{aligned} \begin{pmatrix} 1 \\ 9 \\ 2 \end{pmatrix} &= 1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} \\ &= 3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - 1 \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} \\ &= 2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix}. \end{aligned}$$

Another problem is that we might be working with more vectors than what is required: if v is a linear combination of $\{v_1, \dots, v_m\}$, then $\text{Span}(v_1, \dots, v_m) = \text{Span}(v_1, \dots, v_m, v)$.

For example, since

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix},$$

we have that

$$\text{Span} \left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} \right) = \text{Span} \left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} \right).$$

The good news is that a vector space is completely determined when we choose a generator set that is linearly independent as well.

1. (IN)FINITE DIMENSIONAL SPACES

If a vector space V is the span of a finite set of vectors then we say that V is **finite dimensional**.

Example. \mathbb{R}^n is finite dimensional vector space.

Example. \mathcal{P}_n is a finite dimensional vector space.

Example. $\mathcal{M}_{m \times n}$ is a finite dimensional vector space for any m and n . Indeed, a generator set is the set of matrices $\{M_{i_0 j_0}\}$, where $M_{i_0 j_0}$ is the matrix whose entry in the (i_0, j_0) spot is one, and all the other entries are zero.

Notice that such generator set for $\mathcal{M}_{m \times n}$ consists of mn matrices.

If the vector space V is not finite dimensional, we say that V is **infinite dimensional**.

Example. The vector space \mathcal{P} consisting of all the polynomials with coefficients in the real numbers is infinite dimensional. A generator set for \mathcal{P} is $\{1, x, x^2, x^3, \dots\}$.

2. BASES

2.1. Definition. A **basis** for a vector space V is a linearly independent subset of V that generates V .

In other words, a basis for V is a set of vectors $\{v_1, \dots, v_n\}$ such that

- (1) $\{v_1, \dots, v_n\}$ is linearly independent.
- (2) $V = \text{Span}(v_1, \dots, v_n)$.

Examples.

- The empty set \emptyset is a basis for $\{0\}$.
- Any linearly independent set S is a basis for $\text{Span}(S)$.
- A basis for \mathbb{R}^n is $\{\vec{e}_1, \dots, \vec{e}_n\}$, called the standard basis.
- A basis for \mathcal{P} is $\{1, x, x^2, \dots, x^n, \dots\}$.
- A basis for \mathcal{P}_n is $\{1, x, x^2, \dots, x^n\}$, called the standard basis.
- A basis for $\mathcal{M}_{m \times n}$ is the set of matrices

$$\{A^{kl} = (a_{ij}) \mid a_{ij} = 1 \text{ if } i = k, j = l, \text{ and } a_{ij} = 0 \text{ otherwise}\},$$

called the standard basis.

For instance, a basis for $\mathcal{M}_{2 \times 3}$ is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Example. We would like to obtain a basis for

$$V = \{(x \ y \ z)^T \mid x = 2y - z\}.$$

We obtain a generator set as before by expressing any vector in V in terms of the condition given in the definition of V by noticing that $(a \ b \ c)^T$ is in V if and only if

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2b - c \\ b \\ c \end{pmatrix} = b \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Thus, $B = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a generator set.

Now, we determine if B is linearly independent: the equation

$$\alpha \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

induces the system

$$\begin{array}{rcl} 2\alpha & - & +\beta = 0 \\ \alpha & & = 0 \\ & & \beta = 0 \end{array}$$

The system has only the trivial solution $\alpha = \beta = 0$. Thus, B is linearly independent. We conclude that B is a basis for V .

Remark. A vector space can have more than one basis. Let B and B' be two linearly independent sets of vectors in V . We have that B and B' are bases for V if

$$\text{Span}(B) = V = \text{Span}(B').$$

In other words, B and B' are bases for V if each element in B is a linear combination of B' , and vice-versa.

Example. The sets $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ are two different bases for \mathbb{R}^2 .

2.2. Main Properties. In what follows we will focus on finite dimensional vectors spaces. Though, the situation for infinite dimensional vectors spaces will be similar.

Theorem. Let V be a vector space. Let $B = \{v_1, \dots, v_n\}$ be a subset of V .

- (1) If any vector in V can be uniquely expressed as a linear combination of B , then B is a basis for V .
- (2) If B is a basis for V , then any vector in V can be uniquely expressed as a linear combination of B .

Consequence. If we fix a basis B for a vector space V we have that:

- Any vector v in V determines a UNIQUE sequence of scalars: the coefficients in its linear combination in terms of B .
- Conversely, any n -tuple of scalars determine a UNIQUE vector in V by taking the numbers in the n -tuple as the coefficients of a linear combination of the vectors in B .

Thus, once we fix a basis B for V , each vector in V can be considered as an n -tuple, where n is the number of vectors in the basis B .

Remark. Later we will see that any vector space with n vectors in a basis can be regarded as \mathbb{R}^n .

2.3. Main Theorem. One of the most important results in Linear Algebra is the following

Theorem. Any vector space has a basis.

Given a vector space V , we can obtain a basis for V either from a generating set or from a linearly independent set.

2.3.1. *Bases From Generator Sets:* If the vector space is the zero vector space $\{0\}$, then we know that the empty set $\{\emptyset\}$ is a basis of $\{0\}$.

In general: Let $S = \{v_1, \dots, v_m\}$ be a generator set for V which is not linearly independent and where none of the vectors in S is the zero vector.

Choose any vector from the set, say v_1 . The set $\{v_1\}$ is linearly independent since v_1 is non-zero. Now consider the set $\{v_1, v_2\}$. If $\{v_1, v_2\}$ is not linearly independent, remove v_2 from it, otherwise keep it.

Continue with the process of adjoining elements from S and removing the vectors which are linear combination of the vectors already considered in the set. Since we started with a finite set of vectors, we will finish eventually.

The set obtained at the end will be a basis for $\text{Span}(v_1, \dots, v_m)$.

Remark. In the process described above we are regarding a basis as the minimum set of vectors required to generate a vector space.

2.3.2. *Extending Bases From a Linearly Independent Set:* We have the following theorem.

Theorem. Let V be a vector space with a basis $B = \{v_1, \dots, v_n\}$, which consists of n vectors. Let $S = \{w_1, \dots, w_m\}$ be a linearly independent set in V , with $m \leq n$. Then we can find vectors $\{u_1, \dots, u_{n-m}\}$ in V such that $\{w_1, \dots, w_m, u_1, \dots, u_{n-m}\}$ is a basis for V .

In particular, any linearly independent set consisting of n vectors will be a basis for V .

Example. Notice that $\{x - 1, x + 1\}$ is a linearly independent set in \mathcal{P}_2 . Since x^2 can not be obtained as a linear combination of the vectors in this set and $\dim \mathcal{P}_2 = 3$, we have that

$$\{x - 1, x + 1, x^2\}$$

is a basis for \mathcal{P}_2 .

Consequences. Let V be a vector space with a basis consisting of n vectors. Then

- (1) Any set of vectors containing more than n vectors must be linearly dependent.

Remark. This property says that a basis will consist of a maximal linearly independent set.

- (2) Let V be a vector space with a basis consisting of n elements. Then any basis of V consists of exactly n vectors. In other words, the number of vectors for each basis of a vector space is the same.

3. DIMENSION

3.1. Definition. Let V be a vector space with a basis consisting of n vectors. The **dimension** of V is the number n . It will be denoted as $\dim V$.

Remark. From the previous remarks we can think of the dimension of a vector space as both, the minimum number of vectors required to generate the vector space, and as the maximum number of vectors that a linearly independent set can consist of.

Examples.

- The dimension of $\{0\}$ is zero.
- The dimension of \mathbb{R}^n is n .
- The dimension of \mathcal{P}_n is $n + 1$.
- The dimension of $\mathcal{M}_{m \times n}$ is mn .
- The dimension of any line in a vector space is 1.
- The dimension of any plane in a vector space is 2.

Theorem. Let W be a subspace of a vector space V . Then we have that $\dim W \leq \dim V$. Moreover, if $\dim W = \dim V$, then $W = V$.

Consequences.

- The only subspaces of \mathbb{R}^2 are: the origin, lines through the origin, and \mathbb{R}^2 .
- The only subspaces of \mathbb{R}^3 are: the origin, lines through the origin, planes through the origin, and \mathbb{R}^3 .

3.2. More Examples. Some extra examples that we might see later are:

- (1) The linear space $\mathcal{D}_n = \{(a_{ij}) \in \mathcal{M}_{n \times n} \mid a_{ij} = 0 \text{ if } i \neq j\}$ of diagonal matrices of $n \times n$ has dimension n .
- (2) The linear space $\mathcal{S}_n = \{A \in \mathcal{M}_{n \times n} \mid A^T = A\}$ of symmetric matrices of $n \times n$ has dimension $\frac{n(n+1)}{2}$.
- (3) The linear space $\mathcal{S}_n = \{(a_{ij}) \in \mathcal{M}_{n \times n} \mid a_{ij} = 0 \text{ if } i > j\}$ of upper triangular matrices of $n \times n$ has dimension $\frac{n(n+1)}{2}$.

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