LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

JOSÉ MALAGÓN-LÓPEZ

Given a finite dimensional vector space V, we would like to know what could be the "minimum" number of vectors required to generate V. We will see that this problem is related to determining in how many ways can the zero vector in V be written as a linear combination of vectors in a given generator set.

1. Linear Dependence and Linear Independence

Let V be a vector space, and consider

$$S = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \mid \boldsymbol{u}_i \neq \boldsymbol{0}, \text{ for all } i = 1, \ldots, m\},$$

a set of vectors in V.

Assume that there is a vector $\boldsymbol{v}$ in $\mathrm{Span}(S)$ that can be written as a linear combination of S in two different ways:

$$\alpha_1 \boldsymbol{u}_1 + \cdots + \alpha_m \boldsymbol{u}_m = \boldsymbol{v} = \beta_1 \boldsymbol{u}_1 + \cdots + \beta_m \boldsymbol{u}_m. \tag{1}$$

Here we are assuming that $\alpha_i \neq \beta_i$ for more that one index $1 \leq i \leq m$.

Notice that the equality (1) above is true if and only if

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{v} - \boldsymbol{v} \\ \boldsymbol{0} &= \alpha_1 \boldsymbol{u}_1 + \cdots + \alpha_m \boldsymbol{u}_m - \beta_1 \boldsymbol{u}_1 + \cdots + \beta_m \boldsymbol{u}_m \\ \boldsymbol{0} &= (\alpha_1 - \beta_1) \boldsymbol{u}_1 + \cdots + (\alpha_m - \beta_m) \boldsymbol{u}_m, \qquad (2) \end{aligned}$$

where $\alpha_i - \beta_i \neq 0$ for more than one index $1 \leq i \leq m$.

In other words: the existence of multiple ways to write a vector $\boldsymbol{v}$ as a linear combination of S (identity (1)), is equivalent to the existence of multiple ways to write the zero vector as a linear combination of S (identity (2)).

1.1. **Definition.** Consider a set of vectors $S = \{u_1, \ldots, u_m\}$ in a vector space V.

- We say that S is **linearly dependent** if there are scalars

$$\alpha_1, \ldots, \alpha_m,$$

  such that

$$0 = \alpha_1 u_1 + \cdots + \alpha_m u_m,$$

  where AT LEAST one of the scalars is different than zero.

- We say that S is **linearly independent** if the ONLY scalars

$$\alpha_1, \ldots, \alpha_m,$$

  satisfying the equation

$$0 = \alpha_1 u_1 + \cdots + \alpha_m u_m$$

  are $\alpha_1 = \cdots = \alpha_m = 0$.

Convention. As a convention, we will state that the empty set $\emptyset$ is linearly independent.

Examples.

- If $v \neq 0$ then $\{v\}$ is linearly independent.
- Any collection of vectors containing the zero vector is linearly dependent.
- Two parallel vectors are linearly dependent.
- Two non-parallel vectors are linearly independent.

Notation. If $0 = \alpha_1 u_1 + \cdots + \alpha_m u_m$, where not all the scalars are zero, we say that the zero vector is a *non-trivial linear combination* of the set $\{u_1, \ldots, u_m\}$.

Remark. Given any set of vectors $S = \{u_1, \ldots, u_m\}$, we know that the zero vector 0 is always a trivial linear combination of S:

$$0 = 0u_1 + \cdots + 0u_m.$$

The question of determining whether S is linearly dependent or linearly independent is the question of determining in how many ways we can write the zero vector 0 as a linear combination of S.

In other words, if we can express the zero vector as a non-trivial linear combination of S then S is linearly dependent, otherwise the set S is linearly independent.

Example. The set of matrices in $\mathcal{M}_{2\times 2}$

$$\left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ 4 & 5 \end{pmatrix} \right\}$$

is linearly dependent since

$$2 \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 5 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Remark. Assume that $S = \{u_1, \ldots, u_m\}$ is a linearly dependent set of non-zero vectors. By definition we have that the zero vector is a non-trivial linear combination of S, say

$$\mathbf{0} = \alpha_1 u_1 + \cdots + \alpha_m u_m.$$

We have that at least two of the scalars are non-zero. Assuming without loss of generality that α_1 is one of them, we can write u_1 as a linear combination of other vectors in S:

$$u_1 = -\frac{\alpha_2}{\alpha_1} u_2 - \ldots - \frac{\alpha_m}{\alpha_1} u_m.$$

In other words: a set S of non-zero vectors, is linearly dependent if and only if at least one of the vectors in S is a linear combination of the other vectors in S.

Example. The set of geometric vectors $\left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ -13 \\ -3 \end{pmatrix} \right\}$ is linearly dependent since

$$-2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -13 \\ -3 \end{pmatrix}.$$

Remark. A collection of vectors is either linearly dependent or linearly independent. Therefore, a set S, of non-zero vectors, is linearly independent if and only if none of the vectors in S is a linear combination of the other vectors in S.

1.2. **Basic Technique.** From the remark above we conclude that the problem of determining if a set of vectors is linearly independent is a question about the existence of a non-trivial linear combination. Thus, it is a problem that can be reduced to solving a system of linear equations.

We will illustrate the technique with some explicit examples.

1.2.1. *Example: Geometric Vectors.* We want to determine if the set of vectors

$$S = \left\{ \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -6 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 8 \end{pmatrix} \right\} \qquad \subseteq \quad \mathbb{R}^3$$

is linearly dependent or linear independent.

We need to find all the possible scalars α_1, α_2 and α_3 for which the following equality holds:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ -6 \\ 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 2 \\ -1 \\ 8 \end{pmatrix}.$$

Such equation induces the system of equations

$$\begin{array}{rcrcrcl} 2\alpha_1 & & & + & 2\alpha_3 & = & 0 \\ 5\alpha_1 & - & 6\alpha_2 & - & \alpha_3 & = & 0 \\ 4\alpha_1 & + & 2\alpha_2 & + & 8\alpha_3 & = & 0 \end{array}$$

The first equation implies $\alpha_1 = -\alpha_3$. Plugging this value in the second equation and solving for α_2 we obtain $\alpha_2 = -\alpha_3$.

Plugging all these values in the last equation we obtain the identity

$$2\alpha_3 = 0.$$

Thus, the UNIQUE solution to such system is

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Therefore, the set S is linearly independent.

1.2.2. *Example: Polynomials.* We want to determine if the set of polynomials

$$S = \left\{ \begin{array}{lclcrcrcr} P(x) & = & & & 3x^2 & + & x & - & 1, \\ Q(x) & = & 6x^3 & & & + & 5x & + & 1, \\ R(x) & = & 4x^3 & - & 7x^2 & + & x & + & 3 \end{array} \right\}$$

is linearly dependent or linear independent.

We need to find all the possible scalars α_1, α_2 and α_3 for which the following equality holds:

$$0 = \alpha_1\, P(x) + \alpha_2\, Q(x) + \alpha_3\, R(x)$$

$$= \alpha_1(3x^2 + x - 1) + \alpha_2(6x^3 + 5x + 1) + \alpha_3(4x^3 - 7x^2 + x + 3)$$

$$= (6\alpha_2 + 4\alpha_3)x^3 + (3\alpha_1 - 7\alpha_3)x^2 + (\alpha_1 + 5\alpha_2 + \alpha_3) + (-\alpha_1 + \alpha_2 + 3\alpha_3).$$

Such equation induces the system

$$\begin{array}{rcrcrcl} & & 6\alpha_2 & + & 4\alpha_3 & = & 0 \\ 3\alpha_1 & & & - & 7\alpha_3 & = & 0 \\ \alpha_1 & + & 5\alpha_2 & + & \alpha_3 & = & 0 \\ -\alpha_1 & + & \alpha_2 & + & 3\alpha_3 & = & 0 \end{array}$$

Solving the system above we get

$$\alpha_1 = 7, \qquad \alpha_2 = -2, \qquad \alpha_3 = 3.$$

In other words,

$$0 = 7\, P(x) - 2\, Q(x) + 3\, R(x).$$

Therefore, the set S is linearly dependent.

1.2.3. *Example: Matrices.* We want to determine if the set of matrices

$$S = \left\{ \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \right\}$$

is linearly dependent or linearly independent.

We need to find all the possible scalars α_1, α_2 and α_3 for which the following equality holds:

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= \alpha_1 \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 & -\alpha_1 \\ 2\alpha_1 & 0 \end{pmatrix} + \begin{pmatrix} 3\alpha_2 & 2\alpha_2 \\ 0 & -\alpha_2 \end{pmatrix} + \begin{pmatrix} \alpha_3 & 0 \\ -\alpha_3 & 2\alpha_3 \end{pmatrix}. \end{aligned}$$

By comparing entry-wise the equation above we get the system

$$\begin{array}{rcrcrcl} \alpha_1 & + & 3\alpha_2 & + & \alpha_3 & = & 0 \\ -\alpha_1 & + & 2\alpha_2 & & & = & 0 \\ 2\alpha_1 & & & - & \alpha_3 & = & 0 \\ & - & \alpha_2 & + & 2\alpha_3 & = & 0 \end{array}$$

The third and fourth equations imply

$$\alpha_1 = \frac{\alpha_3}{2}, \qquad \alpha_2 = 2\,\alpha_3.$$

Plugging these values in the second equation we obtain

$$0 = -\alpha_1 + 2\alpha_2 = \left(\frac{\alpha_3}{2}\right) + 2(2\,\alpha_3) = \frac{13}{3}\,\alpha_3.$$

Thus, $\alpha_3 = 0$. If we plug this value in the first equation we can check that the equality holds, so the UNIQUE SOLUTION to the system is

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Therefore, the set S is linearly independent.

1.2.4. *Example: Functions.* When we want to determine if a set of functions $S \subset \mathcal{F}(D, R)$ is linearly independent, it is not always possible to restate the question in terms of a system of linear equations.

This could happen either because functions are not determined by a sequence of components or entries, or we do not have a generating set. But there are ways to approach the problem.

For example, if we want to determine whether

$$S = \{\sin(x), \cos(x)\} \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$$

is linearly independent or dependent.

We need to find all the possible scalars α and β for which the following equality holds:

$$\alpha \sin(x) + \beta \cos(x) = \boldsymbol{0}.$$

Evaluating the equality above at $x = 0$ we get

$$0 = \boldsymbol{0}(0) = \alpha \sin(0) + \beta \cos(0) = \alpha\,(0) + \beta\,(1) = \beta.$$

Thus, β must be zero.

Evaluating the equality above at $x = \pi/2$ we get

$$0 = \boldsymbol{0}(0) = \alpha \sin(\pi/2) + \beta \cos(\pi/2) = \alpha\,(1) + \beta\,(0) = \alpha.$$

Thus, α must be zero. We have shown that

$$\alpha = \beta = 0.$$

Therefore, $S = \{\sin(x), \cos(x)\}$ is linearly independent.

1.3. **Some Facts.** We need to know how much control we have with regards to adding vectors to linearly independent sets, or removing vectors from a linearly dependent set.

Facts. Let $S_1 \subset S_2$ be two non-empty sets of vectors in a vector space V. Then

(1) If S_1 is linearly dependent, then S_2 is linearly dependent.

(2) If S_2 is linearly independent, then S_1 is linearly independent.

(3) If $\{v_1, \ldots, v_m\}$ is a linearly independent set, and v is a vector not in $\text{Span}(v_1, \ldots, v_m)$, then $\{v_1, \ldots, v_m, v\}$ is linearly independent.

(4) If $\{v_1, \ldots, v_m\}$ is a linearly independent set, and v is a vector in $\text{Span}(v_1, \ldots, v_m)$, then $\{v_1, \ldots, v_m, v\}$ is linearly dependent.

Example. We want to determine if

$$S = \{\sin(x), \cos(x), e^x\} \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$$

is linearly independent.

In the example above we saw that $\{\sin(x), \cos(x)\}$ is linearly independent. Thus, S will be linearly independent only if the function e^x is not in $\text{Span}(\sin(x), \cos(x))$.

So, we need to find if there are scalars α and β for which

$$e^x = \alpha \sin(x) + \beta \cos(x).$$

If we consider $x = 0$, we would have

$$\begin{aligned} 1 = e^0 &= \alpha \sin(0) + \beta \cos(0) \\ &= 0 + \beta. \end{aligned}$$

Thus, β should be equal to one.

If we consider $x = \pi/2$ we would have

$$\begin{aligned} e^{\pi/4} &= \alpha \sin(\pi/2) + \cos(\pi/2) \\ &= \alpha + 0. \end{aligned}$$

Thus, α should be equal to $e^{\pi/4}$.

Hence, IF e^x is a linear combination of $\sin(x)$ and $\cos(x)$, it would be
$$e^x = e^{\pi/2}\sin(x) + \cos(x).$$

But this IS NOT TRUE since at $x = \pi$ we have

$$e^\pi = e^{\pi/2}\sin(\pi) + \cos(\pi) = -1,$$

and this IS NOT POSSIBLE because $e^x > 0$ for any x.

Hence, e^x is not a linear combination of $\{\sin(x), \cos(x)\}$. Therefore,
$$\{\sin(x), \cos(x), e^x\}$$
is linearly independent.

E-mail address: `jmalagon@uottawa.ca`