SUBSPACES AND SPANS

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In a vector space we have two basic operations: addition and scalar multiplication. Linear algebra is about the study of the objects that are completely described in terms of such operations. Specifically, we will pay attention to:

1) All the elements in a vector space V that can be written in terms of the two basic operations. This will lead to the notion of linear combination.

2) All the subsets of a vector space V that inherit a structure of vector space from the structure given in V. This will lead to the notion of subspace.

1. Subspaces

A subset W of a vector space V is called a **subspace of** V if W is a vector space with the operations of addition and scalar multiplication given on V.

Example. A vector space V is a subspace of V.

Example. If $\mathit{0}$ is the zero vector of V, then $\{\mathit{0}\}$ is a subspace of V. We say that $\{\mathit{0}\}$ is the **zero subspace**.

Example. The real numbers are a subspace of the complex numbers.

Example. Any line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

Example. Any line or plane through the origin in $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$.

Note. Any line or plane not passing through the origin are not subspaces. The reason is that they will not be closed under addition.

Remark. It is possible that a subset S of a vector space V could have a structure of a vector space under operations different from the operations of addition and scalar product of V. In such case, S is not a subspace of V. As an example of this situation, consider the set S of all the 2×2 matrices of the form

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$$

where a and b are real numbers. We have that S is a vector space under

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} + \begin{pmatrix} a' & 1 \\ 1 & b' \end{pmatrix} = \begin{pmatrix} a + a' & 1 \\ 1 & b + b' \end{pmatrix} \quad \text{and} \quad \alpha \star \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} = \begin{pmatrix} \alpha a & 1 \\ 1 & \alpha b \end{pmatrix}.$$

But we have that S is not a vector space under the standard operations for $\mathcal{M}_{2\times 2}$. Since the standard operations for $\mathcal{M}_{2\times 2}$ are different from the operations defined above, we have that S is not a subspace of $\mathcal{M}_{2\times 2}$.

1.1. **First Test.** If we take a look at the axioms defining a vector space we notice that the only axioms that we need to check to determine if a subset W of a vector space V is a subspace are:

- W is closed under addition;
- there exists an additive identity;
- there exists an additive inverse;
- W is closed under scalar multiplication.

Now, let $\boldsymbol{w}$ be any vector in $W \subseteq V$. If W is closed under scalar multiplication then $-\boldsymbol{w} = (-1)\boldsymbol{w}$ is in W. In other words, we do not need to check the existence of additive inverse in W. Summarizing, we have obtained:

Theorem 1. Let V be a vector space. Let W be a non-empty subspace of V. Then W is a subspace of V if and only if

(1) The zero vector in V is in W.

(2) W is closed under addition.

(3) W is closed under scalar multiplication.

Example. The set $\{(x \quad y)^T \mid x+y=0\}$ is a subspace of $\mathbb{R}^2$ under the standard operations. Indeed, notice that

- $(0 \quad 0)^T$ is in W: we just observe that $(0 \quad 0)^T$ satisfies the equation defining W, $0+0=0$. Thus $(0 \quad 0)^T \in W$.
- W is closed under addition: given $(a \quad b)^T$ and $(a' \quad b')^T$ in W, we have that
  $$a+b=0 \qquad a'+b'=0.$$
  Now, since $(a \quad b)^T + (a' \quad b')^T = (a+a' \quad b+b')^T$, from the equation above we conclude that W is closed under addition:
  $$(a+a')+(b+b') = (a+b)+(a'+b') = 0+0 = 0.$$
- W is closed under scalar multiplication: given any scalar α, and any vector $(a \quad b)^T$ in W, we have that $\alpha(a \quad b)T = (\alpha a \quad \alpha b)T$. We conclude that W is closed under scalar multiplication:
  $$\alpha a + \alpha b = \alpha(a+b) = \alpha(0) = 0.$$

Example. The set $S = \{(x \quad y)^T \mid x+y=1\}$ is not a subspace of $\mathbb{R}^2$. Indeed, the vectors $(1 \quad 1)^T$ and $(1 \quad 1)^T$ are in S, but the vector

$$(1 \quad 1)^T + (1 \quad 1)^T = (2 \quad 2)^T$$

is not in S, so S is not closed under addition.

Example. The set $S = \{(x \quad y)^T \mid xy=0\}$ is not a subspace of $\mathbb{R}^2$. Indeed, the vectors $(1 \quad 0)^T$ and $(0 \quad 1)^T$ are in S, but the vector

$$(1 \quad 0)^T + (0 \quad 1)^T = (1 \quad 1)^T$$

is not in S since $1 \cdot 1 = 1 \neq 0$. Then S is not closed under addition.

Example. The set $S = \{(x \quad y)^T \mid x+y \geq 0\}$ is not a subspace of $\mathbb{R}^2$. Indeed, the vector $(1 \quad 1)^T$ is in S, but the vector

$$(-1)(1 \quad 1)^T = (-1 \quad -1)^T$$

is not in S, so S is not closed under scalar multiplication.

Example. Let $\mathcal{P}$ be the vector space of all the polynomials with coefficients in the real numbers under the standard operations.

Let $\mathcal{P}_n$ denote the set of all the polynomials of degree at most n. Since

$$\deg(\alpha P(x)) \geq \deg(P(x))$$

and

$$\deg(P(x) + Q(x)) \leq max\{\deg(P(x)), \deg(Q(x))\},$$

it follows that $\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

Example. Let $\mathcal{M}_n$ be the vector space of all the $n \times n$ matrices under the standard operations. A square matrix (a_{ij}) is called a **diagonal matrix** if $a_{ij} = 0$ for all $i \neq j$.

Let $\mathcal{D}_n$ denote the set of all the $n \times n$ diagonal matrices. Clearly the $n \times n$ zero matrix is diagonal, and since the standard addition and scalar multiplication for matrices are defined entry-wise we have that $\mathcal{D}_n$ is a subspace of $\mathcal{M}_n$.

Example. Let $\mathcal{M}_{m\times n}$ be the vector space of all the $m \times n$ matrices under the standard operations. The **transpose** A^T of an $m \times n$ matrix A is the $n \times m$ matrix obtained from interchanging the columns with the rows. In other words, the i-th row of A^T corresponds to the i-th column of A.

Let $\mathcal{Sym}_n$ be the set of all the $n\times n$ matrices A such that $A^T = A$. Clearly the $n \times n$ zero matrix is in $\mathcal{Sym}_n$, and since the standard addition and scalar multiplication for matrices are defined entry-wise we have that $\mathcal{Sym}_n$ is a subspace of $\mathcal{M}_n$.

Theorem 2. Any intersection of subspaces is a subspace.
Proof. Let $W = \cap_{i\in I} W_i$ be an intersection of subspaces of a vector space V (each W_i is a subspace of V). Let $\boldsymbol{0}$ be the zero vector in V. Since each W_i is a subspace of V, $\boldsymbol{0}$ is in each W_i. Thus, $\boldsymbol{0}$ is in W.

Given $\boldsymbol{u}$ and $\boldsymbol{u}'$ in W, we have that $\boldsymbol{u}$ and $\boldsymbol{u}'$ is in each W_i, so $\boldsymbol{u} + \boldsymbol{u}'$ is in each W_i. Thus $\boldsymbol{u} + \boldsymbol{u}'$ is in W.

Given any real number α, and any $\boldsymbol{u}$ in W, we have that $\boldsymbol{u}$ is in each W_i, so $\alpha\boldsymbol{u}$ is in each W_i. Thus, $\alpha\boldsymbol{u}$ is in W. QED.

2. LINEAR COMBINATIONS

Our objects of study can be described in terms of linear combinations, so most of the problems to be solved in this course will be reduced to a question about the existence (or not existence) of a linear combination. Without further preamble, we define what a linear combination is.

Let V be a vector space. Let S be a non-empty subset of vectors of V. We say that a vector $\boldsymbol{v}$ in V is a **linear combination** of elements of S if there is a *finite* number of elements

$$\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$$

in S and scalars

$$\alpha_1, \ldots, \alpha_m$$

such that

$$\boldsymbol{v} = \alpha_1 \boldsymbol{u}_1 + \cdots + \alpha_m \boldsymbol{u}_m.$$

The scalars in the expression above are called the **coefficients** of the linear combination.

Example. The zero vector $\boldsymbol{0}$ in a vector space V is a linear combination of any non-empty collection S of vectors in V. Indeed, for any $\boldsymbol{u}$ in S, we have that $\boldsymbol{0} = 0\boldsymbol{u}$.

Example. Any vector $\vec{v} = (v_1 \quad v_2)^T$ in $\mathbb{R}^2$ is a linear combination of the standard unit vectors $\vec{e}_1$ and $\vec{e}_2$:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Example. The vector $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$ is a linear combination of $\left\{ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} \right\}$:

$$\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}.$$

Remark. Notice that in order to say that a vector is a linear combination of a given set of vectors, you need to provide (explicitly) the scalars in the linear combination.

2.1. **Second Test.**

Now we can describe a subspace in terms of linear combinations.

Theorem 3. A non-empty subset W of a vector space is a subspace if and only if any linear combination of vectors in W is again in W.

This theorem simplified the process of determining whether a given subset of a vector space is a subspace.

Example. The set $W = \{(x \quad y)^T \mid x = y\}$ is a subspace of $\mathbb{R}^2$ under the standard operations.

Indeed, given any scalars α and β, and any vectors $(a \quad b)^T$ and $(a' \quad b')^T$ in W we have that

$$\alpha \begin{pmatrix} a \\ b \end{pmatrix} + \beta \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \alpha a + \beta a' \\ \alpha b + \beta b' \end{pmatrix}.$$

Since $a = b$ and $a' = b'$ implies that $\alpha a + \beta a' = \alpha b + \beta b'$, we conclude that W is a subspace of $\mathbb{R}^2$.

Example. The set $W = \{(x \quad y)^T \mid x + y = 0\}$ is a subspace of $\mathbb{R}^2$ under the standard operations. Notice that we already used Test 1 to see that W is a subspace. Now we will use Test 2.

Given any scalars α and β, and any vectors $(a \quad b)^T$ and $(a' \quad b')^T$ in W we have that

$$\alpha \begin{pmatrix} a \\ b \end{pmatrix} + \beta \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \alpha a + \beta a' \\ \alpha b + \beta b' \end{pmatrix}.$$

Since $a + b = 0$ and $a' + b' = 0$ implies that

$$\begin{aligned} (\alpha a + \beta a') + (\alpha b + \beta b') &= (\alpha a + \alpha b) + (\beta a' + \beta b') \\ &= \alpha(a + b) + \beta(a' + b') \\ &= \alpha 0 + \beta 0 = 0, \end{aligned}$$

we conclude that W is closed under linear combinations, so it is a subspace of $\mathbb{R}^2$.

Example. The set $W = \{f : \mathbb{R} \to \mathbb{R} \mid f(1) - f(0) = 0\}$ is a subspace of $\mathscr{F}(\mathbb{R}, \mathbb{R})$.

Indeed, Let α and β be any scalars, and let f and g be any functions in W. This means that $f(1) - f(0) = 0$ and $g(1) - g(0) = 0$. Thus

$$\begin{aligned}(\alpha f - \beta g)(1) + (\alpha f - \beta g)(0) &= \alpha f(1) - \beta g(1) + \alpha f(0) - \beta g(0) \\ &= \alpha(f(1) - f(0)) + \beta(g(1) - g(0)) \\ &= \alpha 0 + \beta 0 = 0.\end{aligned}$$

Therefore, W is a subspace of $\mathscr{F}(\mathbb{R}, \mathbb{R})$.

2.2. **Basic Technique.** A general principle is that the problem of determining whether or not a vector $\mathbf{v}$ is a linear combination of other vectors S, is at the end a problem of whether or not a system of linear equations has a solution. Moreover, the solutions of such system will give us the coefficients of the linear combination.

How to set up such a system of linear equations depends on the nature of the vector space in which the vectors are being considered.

We see how this technique works in two explicit examples.

2.2.1. *Example: Geometric Vectors.* Consider the set vectors in $\mathbb{R}^3$

$$S = \left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \right\}.$$

We will determine if any of the vectors $\begin{pmatrix} 5 \\ -13 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ -13 \\ -3 \end{pmatrix}$ are a linear combination of the vectors in S. In general, any vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ would be a linear combination of the vectors in S if there are scalars α and β such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -\alpha + \beta \\ 2\alpha - 3\beta \\ -\beta \end{pmatrix}.$$

Since two vectors are equal if and only if their components are equal, comparing component-wise we obtain the system of linear equations

$$\begin{array}{rcrcl} -\alpha & + & \beta & = & x \\ 2\alpha & - & 3\beta & = & y \\ & - & \beta & = & z \end{array}$$

Thus, we have reduced the problem of determining if a vector is a linear combination of S to the existence of a solution for the system above.

- Applying the procedure above to the vector $\begin{pmatrix} 5 \\ -13 \\ -3 \end{pmatrix}$ we obtain the system

  $$\begin{array}{rcrcr} -\alpha & + & \beta & = & 5 \\ 2\alpha & - & 3\beta & = & -13 \\ & - & \beta & = & -3 \end{array}$$

  The last equation implies $\beta = 3$. Plugging this value in the second equation gives us $\alpha = 2$.

  We see that the system has a solution by plugging these values in the first equation and verifying that the equality holds. Thus, we had obtained the linear combination

  $$-2\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -13 \\ -3 \end{pmatrix}.$$

- Applying the procedure above to the vector $\begin{pmatrix} -5 \\ -13 \\ -3 \end{pmatrix}$ we obtain the system

  $$\begin{array}{rcrcr} -\alpha & + & \beta & = & -5 \\ 2\alpha & - & 3\beta & = & -13 \\ & - & \beta & = & -3 \end{array}$$

  The last equation implies $\beta = 3$. Plugging this value in the second equation gives us $\alpha = 2$.

  We see that the system has no solution when we plug these values in the first equation and get $5 = -5$, which is not true. This implies that $\begin{pmatrix} -5 & -13 & -3 \end{pmatrix}^T$ is not a linear combination of the vectors in S.

2.2.2. *Example: Polynomials.* Consider the polynomials

$$v_1(x) = 2x^2 + x \quad \text{and} \quad v_2(x) = -x^2 + 1$$

in $\mathcal{P}_2$. We will determine if the polynomials

$$P(x) = x^2 + 2x - 1 \quad \text{and} \quad Q(x) = 7x^2 + 3x - 1$$

are linear combinations of v_1 and v_2.

In general, a quadratic polynomial $ax^2 + bx + c$ is a linear combination of v_1 and v_2 if there are scalars α and β such that

$$\begin{aligned} ax^2 + bx + c &= \alpha v_1 + \beta v_2 = \alpha(2x^2 + x) + \beta(-x^2 + 1) \\ &= (2\alpha - \beta)x^2 + \alpha x + \beta. \end{aligned}$$

This induces the system

$$\begin{array}{rcrcr} 2\alpha & - & \beta & = & a \\ \alpha & & & = & b \\ & & \beta & = & c \end{array}$$

- Applying this to the case of $P(x) = x^2 + 2x - 1$ we obtain the system
$$\begin{array}{rcrcr} 2\alpha & - & \beta & = & 1 \\ \alpha & & & = & 2 \\ & & \beta & = & -1 \end{array}$$
The last two equations imply $\alpha = 2$ and $\beta = -1$. Plugging these values in the first equation we get $5 = 1$, which is false. Thus the system has no solution. This implies that $P(x)$ is not a linear combination of v_1 and v_2.
- Applying this to the case of $P(x) = x^2 + 2x - 1$ we obtain the system
$$\begin{array}{rcrcr} 2\alpha & - & \beta & = & 7 \\ \alpha & & & = & 3 \\ & & \beta & = & -1 \end{array}$$
The last two equations imply $\alpha = 3$ and $\beta = -1$. Plugging these values in the first equation we get $7 = 7$, so the system has a solution. We have obtained the linear combination
$$Q(x) = 3v_1 - v_2.$$

3. SPAN

Let S be a non-empty subset of a vector space V. The **span** of S, denoted as $\mathbf{Span}(S)$, is the set of *all* the linear combinations of elements in S.

Example. If $S = \emptyset$ is the empty set, we will have the convention $\text{Span}(\emptyset) = \{\mathit{0}\}$.

Example. We have that $\mathbb{R}^n = \text{Span}(\vec{e}_1, \ldots, \vec{e}_n)$.

Example. We have $\mathcal{M}_{2,2} = \text{Span}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$.

Example. Given any non-zero vector $\vec{v}$ in $\mathbb{R}^n$, the set $\text{Span}(\vec{v})$ defines a line through the origin.

Remark. If $S_1 \subseteq S_2$ are two subsets of a vector space V then we have that $\text{Span}(S_1) \subseteq \text{Span}(S_2)$.

3.1. **Third Test.** Another way to test if a subset is a subspace is based on the following theorem.

Theorem 4. Let W be a subset of a vector space V. Then W is a subspace if and only if $\text{Span}(W) = W$.

Example. We want to determine if the subset

$$S = \{(x \quad y \quad z)^T \mid x = 2y,\ z = -y\}$$

is a subspace of $\mathbb{R}^3$. From the conditions defining S we have that for each vector in S there is no restriction on which values its second component can take, so we can think of the second component as a parameter t: any vector $(a \quad b \quad c)^T$ in S looks like

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2t \\ t \\ -t \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

for some real number t. From the previous theorem we conclude that S is a subspace since $S = \text{Span}\left((2 \quad 1 \quad -1)^T\right)$.

Example. We want to determine if the subset

$$S = \left\{ \begin{pmatrix} x \\ x+y \\ x-2y \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^3$. Since any element in S is of the form

$$\begin{pmatrix} x \\ y \\ x-2y \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ -2y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix},$$

we conclude that

$$S = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right)$$

Therefore, S is a subspace of $\mathbb{R}^3$.

4. GENERATOR SET

If a vector space V is of the form $V = \operatorname{Span}(S)$, for some subset S of V, we say that S **generates** V. In this situation, we also say that the elements of S **generate** (or **span**) V.

Example. A generator set for $\mathbb{R}^2$ is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. Indeed, notice that any vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathbb{R}^2$ can be written as

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a-b) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Example. A generator set for $\mathbb{R}^n$ is $\{\vec{e}_1, \ldots, \vec{e}_n\}$.

Example. A generator set for $\mathcal{M}_{2\times 3}$ is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Example. A generator set for the vector space $\mathcal{P}_n$ consisting of all the polynomials with coefficients in the real numbers of degree at most n is $\{1, x, x^2, \ldots, x^{n-1}, x^n\}$.

Example. Let $U = \text{Span}\left(\begin{pmatrix}1\\2\end{pmatrix}\right)$. Consider the set

$$W = \left\{ \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 \mid \begin{pmatrix}x\\y\end{pmatrix} \bullet \vec{u} = 0, \quad \text{for all } \vec{u} \text{ in } U \right\},$$

the set of all orthogonal vectors to any vector in U. We have that W a subspace of $\mathbb{R}^2$.

Indeed, since any vector in W is such that

$$0 = \begin{pmatrix}x\\y\end{pmatrix} \bullet \alpha \begin{pmatrix}1\\2\end{pmatrix} = \alpha(x + 2y).$$

Thus, for any vector $\begin{pmatrix}x\\y\end{pmatrix}$ in W we must have that $x = -2y$.

In other words, any vector in W is of the form

$$\begin{pmatrix}-2y\\y\end{pmatrix} = y\begin{pmatrix}-2\\1\end{pmatrix}.$$

Hence,

$$W = \text{Span}\left(\begin{pmatrix}-2\\1\end{pmatrix}\right).$$

Therefore, W is a subspace of $\mathbb{R}^2$.

Notice that

- U is a line passing through the origin with direction vector $\begin{pmatrix}1\\2\end{pmatrix}$.
- W is a line passing through the origin with direction vector $\begin{pmatrix}-2\\1\end{pmatrix}$.

Remark. A vector space can have more than one generator set. For example, the sets of vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad \text{and} \qquad \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

are both generators sets of $\mathbb{R}^2$.

Theorem 5. Let V be a vector space. Then S_1 and S_2 are both generating sets of V if and only if

(1) Any vector in S_1 is a linear combination of the vectors in S_2.

(2) Any vector in S_2 is a linear combination of the vectors in S_1.

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