VECTORS SPACES OVER $\mathbb{R}$

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The algebraic structure that we saw in $\mathbb{R}^2$ and $\mathbb{R}^3$ (addition, scalar multiplication and the way these operations are compatible) can be extended to $\mathbb{R}^n$, where n is any positive integer. A noteworthy observation is that such structure is not exclusive of $\mathbb{R}^n$.

The notion of vector space is the abstraction of such structure, and consists of the minimum properties required in order to work with such objects.

1. Formal Definition of a Vector Space Over $\mathbb{R}$

In order to define a vector space we require two sets: a collection of "numbers" that will play the role of scalars, and the collection of "vectors". In this course, the **scalars** will be the real numbers.

Definition. A **vector space over** $\mathbb{R}$ is a set V endowed with two operations: *addition* $+$, and *scalar multiplication* $\cdot$, such that

(1) *V is closed under addition.* For all $\boldsymbol{v}$ and $\boldsymbol{w}$ in V, there is a unique $\boldsymbol{v}+\boldsymbol{w}$ in V.
(2) *Commutativity.* For all $\boldsymbol{v}$ and $\boldsymbol{w}$ in V, $\boldsymbol{v}+\boldsymbol{w}=\boldsymbol{w}+\boldsymbol{v}$.
(3) *Associativity.* For all $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ in V, $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$.
(4) *Existence of Additive Identity.* There is an element $\boldsymbol{0}$ in V such that for all $\boldsymbol{v}$ in V, $\boldsymbol{v}+\boldsymbol{0}=\boldsymbol{v}$.
(5) *Existence of Additive Inverse.* For all $\boldsymbol{v}$ in V there is an element $\boldsymbol{v}'$ such that $\boldsymbol{v}+\boldsymbol{v}'=\boldsymbol{0}$.
(6) *V is closed under scalar multiplication.* For all $\boldsymbol{v}$ in V and all α in $\mathbb{R}$, there is a unique $\alpha\cdot\boldsymbol{v}$ in V.
(7) For all $\boldsymbol{v}$ in V, $1\cdot\boldsymbol{v}=\boldsymbol{v}$.
(8) For all $\boldsymbol{v}$ in V, and all α and β in $\mathbb{R}$, $(\alpha\beta)\cdot\boldsymbol{v}=\alpha\cdot(\beta\cdot\boldsymbol{v})$.
(9) For all $\boldsymbol{v}$ and $\boldsymbol{w}$ in V, and all α in $\mathbb{R}$, $\alpha\cdot(\boldsymbol{v}+\boldsymbol{w})=\alpha\cdot\boldsymbol{v}+\alpha\cdot\boldsymbol{w}$.
(10) For all $\boldsymbol{v}$ in V and all α and β in $\mathbb{R}$, $(\alpha+\beta)\cdot\boldsymbol{v}=\alpha\cdot\boldsymbol{v}+\beta\cdot\boldsymbol{w}$.

In what follows, we will just say that V is a vector space. Keeping the analogy with $\mathbb{R}^n$, the elements of a V are called **vectors**.

CAUTION! Do not confuse the current use of the word vector with the usual notion of an object defined by a direction and a magnitude. From now on, the word vector will be used to describe any element in a vector space.

The element $\boldsymbol{v} + \boldsymbol{w}$ is called the **sum** of $\boldsymbol{v}$ and $\boldsymbol{w}$. The element $\alpha \cdot \boldsymbol{v}$ is called the **scalar product of $\boldsymbol{v}$ by** α.

1.1. **Example: n-th Spaces.** The n-th space $\mathbb{R}^n$ is the collection of all the ordered n-truples of the form $(v_1, v_2, \ldots, v_n)$, where each v_i is a real number.

We will denote as

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

the vector defined by the point $(v_1, v_2, \ldots, v_n)$. Sometimes we will use the "transpose" notation

$$\vec{v} = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \end{pmatrix}^T.$$

Let $\vec{v} = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}^T$ and $\vec{w} = \begin{pmatrix} w_1 & \ldots & w_n \end{pmatrix}^T$ be two vectors in $\mathbb{R}^n$. Let α be a scalar.

The set $\mathbb{R}^n$ have a structure of vector space with addition and scalar multiplication defined as follows:

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix},$$

$$\alpha \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{pmatrix}.$$

The operations defined above will be called the **standard operations** in $\mathbb{R}^n$. In this case, the additive identity is given by the zero vector

$$\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Remark. Notice that $n = 1$ says that $\mathbb{R}$ is a vector space over itself.

Remark. All the basic operations defined for $\mathbb{R}^2$ can be extended to $\mathbb{R}^n$.

Let $\vec{v} = (v_1, \ldots, v_n)^T$ and $\vec{w} = (w_1, \ldots, w_n)^T$ be any two vectors in $\mathbb{R}^n$. Then:

- Dot Product: $\vec{v} \bullet \vec{w} = v_1 w_1 + \cdots + v_n w_n$.
- Norm: $\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$.
- Two vectors are **orthogonal** if their dot product is zero.
- Two vectors $\vec{v}$ and $\vec{w}$ are **collinear** if there is a non-zero scalar α such that $\vec{v} = \alpha \vec{w}$.
  If such α does not exists, we say that $\vec{v}$ and $\vec{w}$ are **non-collinear**.
- If θ is the angle between $\vec{v}$ and $\vec{w}$, we have that

$$\cos(\theta) = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|},$$

- The **projection of $\vec{v}$ along $\vec{w}$** is given as

$$\operatorname{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \bullet \vec{w}}{\vec{w} \bullet \vec{w}} \vec{w}.$$

Remark. Notice that we did not talk about cross product. The notion of cross product is exclusive of $\mathbb{R}^3$.

1.2. **Example: Complex Numbers.** Recall that given any two complex numbers $z = a + b\imath$ and $z' = a' + b'\imath$, the product is given as

$$z'z = (a'a - b'b) + (a'b + ab')\imath.$$

So when z' is a real number $z' = a' + 0\imath$, we get

$$a'z = a'a + a'b\imath.$$

The complex numbers $\mathbb{C}$ are a vector space when addition is the usual addition of complex numbers, and scalar multiplication is given by the usual product of a complex number by a real number.

In this case, the additive identity is the (real/complex) zero number $0 + 0\imath$.

1.3. **Example: Matrices.** An $m \times n$ **matrix** with entries in $\mathbb{R}$ is an array of the form

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where each entry a_{ij}, for $1 \leq i \leq m$, $1 \leq j \leq n$, is a real number.

An $n \times n$ matrix is called **square**. Two matrices are **equal** if all their corresponding entries are equal.

The entries $a_{i1}, a_{i2}, \ldots, a_{in}$ compose the i**-th row** of A.

The entries $a_{1j}, a_{2j}, \ldots, a_{mj}$ compose the j**-th column** of A.

If $n = m$, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the **diagonal entries** of A.

The set of all the $m \times n$ matrices, which we will denote by $\mathcal{M}_{m\times n}$, is a vector space under the operations:

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \qquad \text{and} \qquad \alpha(a_{ij}) = (\alpha a_{ij}).$$

In this case, the additive identity is given by the **zero matrix**, the matrix whose all entries are zero.

1.4. **Example: Functions.** Let $\mathcal{F}(D, R)$ denote the set of all the real-valued functions whose domain is D and whose range is R. Two functions f and g in $\mathcal{F}(D, R)$ are equal if $f(x) = g(x)$ for any x in D.

The set $\mathcal{F}$ is a vector space under the operations:

$$(f+g)(x) = f(x) + g(x) \qquad \text{and} \qquad (\alpha f)(x) = \alpha\,(f(x))\,,$$

where f and g are in $\mathcal{F}$. In this case, the additive identity is given by the **zero function** $\mathcal{0}(x) = 0$.

1.5. **Example: Polynomials.** A **polynomial** with coefficients in $\mathbb{R}$ is an expression of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where each a_i is a real number and called the **coefficient** of x^i.

Two polynomials $P(x)$ and $Q(x)$ are equal if all their corresponding coefficients are equal, in other words, if the coefficient of x^i in $P(x)$ is equal to the coefficient of x^i in $Q(x)$, for all i.

The **degree** of a polynomial $P(x)$ is the largest exponent of the unknown x for which the coefficient is non-zero.

The **zero polynomial** $\mathcal{0}(x)$ is the polynomial where all the coefficients are zero.

The set of all the polynomials with coefficients in the real numbers, which is denoted by $\mathcal{P}$, is a vector space under the operations defined as follows: given any two polynomials $P(x) = a_n x^n + \cdots a_1 x + a_0$ and $Q(x) = b_m x^m + \cdots + b_1 x + b_0$, without lost of generality we can assume that $n > m$. Define $b_{m+1} = b_{m+2} = \cdots b_n = 0$.

The addition is given by

$$P(x) + Q(x) = (a_n + b_n)x^n + \cdots (a_1 + b_1)x + (a_0 + b_0).$$

For any scalar α, define

$$\alpha P(x) = (\alpha a_n)x^n + \cdots (\alpha a_1)x + (\alpha a_0).$$

The additive identity is given by the zero polynomial.

1.6. **Example: Sequences.** A **sequence** is a function σ from the positive integers to the real numbers. If we set $\sigma(n) = a_n$, then the sequence definded by σ is the collection of numbers

$$\{a_n\}.$$

Let Seq denote the set of all the sequences. We have that Seq is a vector space with the operations:

$$\{a_n\} + \{b_n\} = \{a_n + b_n\},$$

and

$$\alpha\{a_n\} = \{\alpha\, a_n\}.$$

The additive identity is given by the **zero sequence** $\{0\}$.

1.7. **NON-Example.** The rational numbers are not a vector space since they are not closed under scalar multiplication.

1.8. **NON-Example.** Consider $\mathbb{R}^2$ under the standard addition and scalar product given by

$$\alpha \star \begin{pmatrix} a \\ b \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \end{pmatrix} & \text{if } \alpha = 0 \\ \\ \begin{pmatrix} \alpha a \\ b/\alpha \end{pmatrix} & \text{if } \alpha \neq 0 \end{cases}$$

Notice that

$$(\alpha + \beta) \star \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a + \beta a \\ b/\left(\alpha + \beta\right) \end{pmatrix}$$

and

$$\alpha \star \begin{pmatrix} a \\ b \end{pmatrix} + \beta \star \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a + \beta a \\ b/\alpha + b/\beta \end{pmatrix}.$$

In general $\frac{b}{\alpha+\beta} \neq \frac{b}{\alpha} + \frac{b}{\beta}$, so we conclude that $\mathbb{R}^2$ is not a vector space under the proposed structure since axiom (10) fails to hold.

1.9. **Properties.** Next, we list the basic properties of a vector space.

Properties 1.

(1) The additive identity is unique.

(2) The additive inverse is unique.

The additive identity sometimes will be referred as the **zero vector**. The additive inverse will be denoted as $-\boldsymbol{v}$.

Properties 2.

(1) $0 \cdot \boldsymbol{v} = \boldsymbol{0}$, for any $\boldsymbol{v}$ in V.

(2) $(-\alpha) \cdot \boldsymbol{v} = \alpha(-\boldsymbol{v}) = -(\alpha \boldsymbol{v})$, for any $\boldsymbol{v}$ in V and any scalar α.

(3) $\alpha \boldsymbol{0} = \boldsymbol{0}$, for any scalar α.

Fact. In a vector space V, the **cancellation law for addition** holds: if $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in V such that $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$, then $\boldsymbol{v} = \boldsymbol{w}$.

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