LINES AND PLANES IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

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1. Case $\mathbb{R}^2$

When we think of $\mathbb{R}^2$ as a collection of points (not vectors), any line is determined by either any two points or a point and its slope.

What happens when we think of $\mathbb{R}^2$ as a collection of vectors?

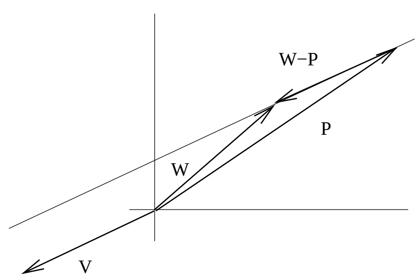
Let $\vec{P}$ be any vector whose end point is on the line ℓ. In such case we say that the line ℓ **passes through** $\vec{P}$.

Note. The line ℓ is determined by the end points of ALL the vectors which the line ℓ passes through.

Let $\vec{v}$ be a vector parallel to ℓ. In such case we say that $\vec{v}$ is a **direction vector** of ℓ.

Note. A direction vector plays the role of the slope in this new setting.

Let ℓ, $\vec{P}$ and $\vec{v}$ be as above. Let $\vec{w}$ be such that ℓ passes through it.



Notice that $\vec{w} - \vec{P}$ is parallel to $\vec{v}$, so there exists a unique $\alpha \neq 0$ such that

$$\alpha\,\vec{v} = \vec{w} - \vec{P}.$$

We conclude that

$$\vec{w} = \vec{P} + \alpha\vec{v} \tag{1}$$

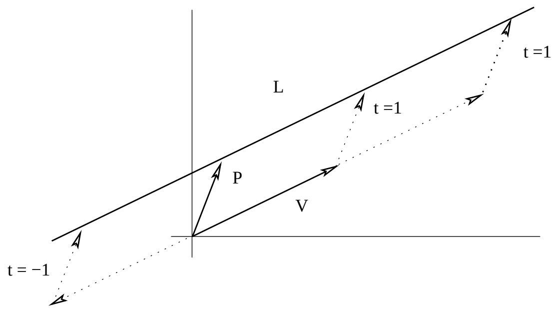
for some scalar α.

1.1. **Vector Equation of a Line.** From the previous discussion we have that for given vectors $\vec{v}$ and $\vec{P}$ so that a line ℓ passes through $\vec{P}$ and $\vec{v}$ is parallel to ℓ, we have that for any vector $\vec{w}$ for which ℓ passes through there is a unique number α so that the equation (1) holds.

Notice that as α runs over all the (real) numbers we can talk about the function

$$\vec{X}(t) = \vec{P} + t\,\vec{v}, \tag{2}$$

where t is an independent variable.



The function above determines the line ℓ by giving us all the vectors for which ℓ passes through. The function (2) is a **vector equation** of ℓ.

Example. The vector equation of the line ℓ passing through $(1, 2)$ and parallel to $(1, 1)$ is

$$\ell : \qquad \vec{X}(t) = (1, 2) + t(1, 1).$$

Thus we can check that ℓ passes through the vector $(6, 7)$ since the equation above hold for $t = 5$:

$$(6, 7) = (1, 2) + 5(1, 1).$$

But ℓ does not pass through the vector $(7, 6)$ since there is no t for which the equation will holds: we should have that

$$7 = 1 + t \qquad \text{and} \qquad 6 = 2 + t.$$

Solving these equations we conclude that $t = 6 = 4$, which is a contradiction, implying that such t does not exists.

Remark. The function $\vec{X}(t) = t\,\vec{v}$ determines a line through the origin. So, if $\vec{P} \neq \vec{0}$ then the function $\vec{X}(t) = \vec{P} + t\,\vec{v}$ is the "translation" of the line $\vec{X}(t) = t\,\vec{v}$ along the vector $\vec{P}$.

Remark. If $y = mx + b$ determines a line ℓ, the direction vector of ℓ is given by $\vec{v} = (1, m)$.

1.2. **Parametric Equation of a Line.** The function (2) is an example of a **parametric representation** of a geometrical object. The independent variable is called the **parameter**. Given a line

$$\ell : \qquad \vec{X}(t) = \vec{P} + t\,\vec{v},$$

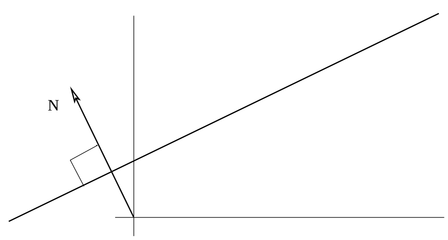
with $\vec{P} = (p_1, p_2)$ and $\vec{v} = (v_1, v_2)$, the components of $\vec{X}(t)$ are given by

$$x_1(t) = p_1 + t\,v_1 \qquad x_2(t) = p_2 + t\,v_2. \tag{3}$$

The equations above are **parametric equations** of ℓ.

1.3. **Normal Equation of a Line.** We would like to represent a line by means of dot product.

We say that a vector $\vec{n}$ is **normal** to a line ℓ if $\vec{n}$ is orthogonal to any direction vector of ℓ.



Let $\vec{n}$ be a vector in standard position and normal to the line

$$\ell : \qquad \vec{X}(t) = \vec{P} + t\,\vec{v}.$$

If ℓ passes through $\vec{w}$ we have that $\vec{P} - \vec{w}$ is a vector direction of ℓ. This implies that $\vec{n} \bullet (\vec{w} - \vec{P}) = 0$. The vectors $\vec{w}$ satisfying this identity determines the line ℓ. The equation

$$\vec{n} \bullet \left(\vec{x} - \vec{P}\right) = 0 \tag{4}$$

is a **normal equation** of the line ℓ.

Example. Working with the example above:

$$\ell : \qquad \vec{X}(t) = (1,2) + t(1,1).$$

Since $(1,1)$ is the direction vector, a normal vector to $(1,1)$ is found by solving:

$$0 = (1,1) \bullet (n_1, n_2) = n_1 + n_2.$$

So we need $n_1 = -n_2$. A solution is the vector $(1,-1)$. Thus, the normal equation to ℓ is

$$\ell : (1,-1) \bullet (\vec{x} - (1,2)) = 0.$$

We check that $(6,7)$ satisfies the equation:

$$(1,-1) \bullet ((6,7) - (1,2)) = (1,-1) \bullet (5,5) = 0.$$

But $(7,6)$ does not satisfies the equation:

$$(1,-1) \bullet ((7,6) - (1,2)) = (1,-1) \bullet (6,4) = 2 \neq 0.$$

1.4. **General Equation of a Line.** If we have that $\vec{n} = (A,B)$ and $\vec{P} = (p_1, p_2)$, and we set $\vec{x} = (x_1, x_2)$, we conclude

$$0 = \begin{pmatrix} A \\ B \end{pmatrix} \bullet \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right) = \begin{pmatrix} A \\ B \end{pmatrix} \bullet \begin{pmatrix} x_1 - p_1 \\ x_2 - p_2 \end{pmatrix} = Ax_1 + Bx_2 - Ap_1 - Bp_2.$$

If we set $C = -Ap_1 - Bp_2$, we obtain the **general equation** of the line

$$\ell : \qquad Ax_1 + Bx_2 + C = 0. \tag{5}$$

2. Case $\mathbb{R}^3$

2.1. **Lines.** In $\mathbb{R}^3$ a line is still determined by two points, so we can proceed as we did for the equations (1) and (2).

Given two vectors $\vec{P}$ and $\vec{v}$ in $\mathbb{R}^3$, the line ℓ passing through $\vec{P}$ and parallel to $\vec{v}$ can be defined as follows:

- (Vector Equation) $\ell : \quad \vec{X}(t) = \vec{P} + t\vec{v}.$

- (Parametric Equations) If $\vec{P} = (p_1, p_2, p_3)$ and $\vec{v} = (v_1, v_2, v_3)$, we have that ℓ is given by

$$x_1(t) = p_1 + tv_1$$
$$x_2(t) = p_2 + tv_2$$
$$x_3(t) = p_3 + tv_3$$

2.2. **Planes.** However, in $\mathbb{R}^3$ it is not longer true that a line is determined by a vector normal to it (we have 3 different directions!).

2.2.1. *Vector and Parametric Equations.* Let $\vec{v}$ and $\vec{w}$ be two vectors non-parallel in $\mathbb{R}^3$. Let s and t denote two parameters. The **vector equation**

$$\mathcal{E}: \qquad \vec{X}(s,t) = s\vec{v} + t\vec{w}$$

defines a **plane** passing through the origin.

Given a vector $\vec{P}$, the equation

$$\mathcal{E}: \qquad \vec{X}(s,t) = \vec{P} + s\vec{v} + t\vec{w} \tag{6}$$

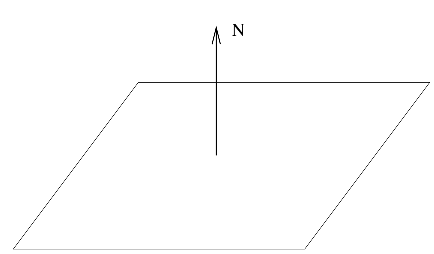
defines the plane passing through $\vec{P}$ and parallel to the plane $s\vec{v} + t\vec{w}$.

If $\vec{P} = (p_1, p_2, p_3)$ and $\vec{w} = (w_1, w_2, w_3)$, from the equation (6) we obtain a **parametric equation**

$$x_1(t) = p_1 + sv_1 + tw_1$$
$$x_2(t) = p_2 + sv_2 + tw_2$$
$$x_3(t) = p_3 + sv_3 + tw_3$$

of the plane $\mathcal{E}$.

2.2.2. *Normal Equation.* We say that a vector $\vec{n}$ in $\mathbb{R}^3$ is **normal** to a plane $\mathcal{E}$ as given by the vector equation (6) if $\vec{n} \bullet (s\vec{v} + t\vec{w}) = 0$.

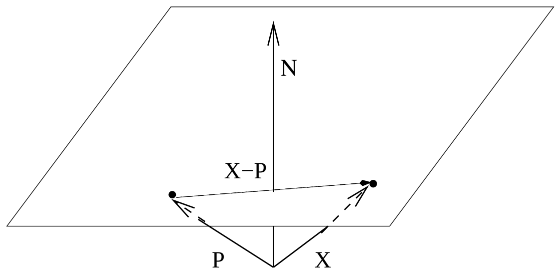


In particular, $\vec{n} \bullet \vec{v} = 0$ and $\vec{n} \bullet \vec{w} = 0$.

Proceeding as for the case of $\mathbb{R}^2$, we get that the **normal equation**

$$\mathcal{E}: \qquad \vec{n} \bullet \left(\vec{X} - \vec{P}\right) = 0$$

defines a plane in $\mathbb{R}^3$, with $\vec{n}$ normal to it.



Note. If $\vec{u}$ is a vector orthogonal to $\vec{n}$, the we say that $\vec{u}$ is parallel to the plane $\mathcal{E}$.

2.2.3. *General Equation.* In this case, a **general equation** of $\mathcal{E}$ is of the form

$$Ax_1 + Bx_2 + Cx_3 = D.$$

Note. Given a plane $\mathcal{E} : Ax_1 + Bx_2 + Cx_3 = D$, the vector (A, B, C) is normal to $\mathcal{E}$.

3. Basic Exercises

To find the relationship between two geometric objects (for example, are they parallel, orthogonal, etc) we will reduce the problem to a question about their direction vectors or their normal vectors.

The problem of finding what is the intersection of two geometric objects will be solved by means of systems of linear equations. In general terms we have

- the objects do not intersect if the system has no solution;
- the intersection set consists of a single vector if the system has a unique solution;
- the intersection set is given by an infinite number of vectors if the system has multiple solutions.

3.1. **Intersection of Lines and Planes.**

3.1.1. *Example.* Consider the lines

$$\begin{array}{llll} \ell_1: & 5x & +3y & = 6 \\ \ell_2: & 2x & & = 18 \end{array}$$

(1) Is ℓ_1 parallel to ℓ_2?
(2) Is ℓ_1 orthogonal to ℓ_2?
(3) What is the intersection set?

Normal vectors associated with ℓ_1 and ℓ_2 are $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ respectively.

For (1): We know that ℓ_1 is parallel to ℓ_2 if and only if there is a non-zero scalar α such that $\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. This is not possible since there is no α such that $3 = 0\alpha$.

Hence, ℓ_1 is not parallel to ℓ_2.

For (2): ℓ_1 is not orthogonal to ℓ_2 since

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 10 \neq 0.$$

For (3): the intersection set is given by ALL the vectors that satisfy simultaneously the equations defining ℓ_1 and ℓ_2.

In other words, we must solve the system

$$\begin{array}{lll} 5x & +3y & = 6 \\ 2x & & = 18 \end{array}$$

The second equation implies $x = 9$, plugging this value in the first equation and solving for y we obtain $y = -13$.

Hence, the intersection set is the single vector $\begin{pmatrix} 9 \\ -13 \end{pmatrix}$.

3.1.2. *Example.* Given the lines

$$\ell_1 : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad \textit{and} \qquad \ell_2 : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + s \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

(1) Is ℓ_1 parallel to ℓ_2?
(2) Is ℓ_1 orthogonal to ℓ_2?
(3) What is the intersection set?

Direction vectors of ℓ_1 and ℓ_2 are $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ respectively.

For (1): we are looking for α such that $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$. This is not possible since by comparing the second components we require $0 = 3$, which is not possible. Hence, ℓ_1 is not parallel to ℓ_2.

For (2): by definition, ℓ_1 is not orthogonal to ℓ_2 since

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = 2 \neq 0.$$

For (3): the intersection set is given by ALL the vectors that satisfy

$$\begin{array}{ccccccc} 1 & + & t & = & 1 & + & 0s \\ 2 & + & 0t & = & 1 & + & 3s \\ 1 & + & 2t & = & -1 & + & s \end{array}$$

The first equation implies $t = 0$. The second equation implies $s = 1/3$. Plugging these values in the third equation gives the identity $1 = -2/3$, which is a contradiction. Thus, the system has no solution, so the two lines do not intersect.

3.1.3. *Example.* Given the planes

$$\begin{array}{llll} \mathcal{E}_1: & x & +2y & +z & = 0 \\ \mathcal{E}_2: & x & -y & -z & = 0 \end{array}$$

(1) Is $\mathcal{E}_1$ parallel to $\mathcal{E}_2$?
(2) What is the intersection set?

Normal vectors of $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ respectively.

For (1): we are looking for α such that $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. This is not possible since the first component requires $1 = \alpha$, and the third component requires $\alpha = -1$. Hence, $\mathcal{E}_1$ is not parallel to $\mathcal{E}_2$.

For (2): We need to solve the system

$$\begin{array}{llll} x & +2y & +z & = 0 \\ x & -y & -z & = 0 \end{array}$$

Second equations gives us $x = y + z$. Plugging this value in the first equation and solving for y we get $y = \frac{-2z}{3}$.

Notice that z can take ANY value, in other words, z plays the role of a parameter. Thus, the intersection set is given by all the vectors of the form

$$\begin{pmatrix} 1/3t \\ -2/3t \\ t \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ 1 \end{pmatrix} t,$$

where t is the parameter.

3.1.4. *Example.* Find the intersection of the line

$$\ell : \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

with the plane

$$\mathcal{E} : x - 2y + 3z = 1.$$

The components of any vector in the intersection must satisfy the two equations. Since the vectors passing through ℓ are of the form

$$\begin{pmatrix} 1-t \\ t \\ 2 \end{pmatrix},$$

by plugging in these values in the equation defining the plane $\mathcal{E}$ we get

$$1 = (1-t) - 2(t) + 3(2) = 7 - 3t.$$

Thus, $t = 2$ gives the intersection set:

$$\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

3.2. **Miscellaneous Problems.**

3.2.1. *Example.* Find a normal equation of the plane $\mathcal{E}$ containing the lines

$$\ell_1 : \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \ell_2 : \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

We find the normal vector to the plane by means of cross product:

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}.$$

Thus,

$$\mathcal{E} : 0 = \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right] \bullet \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix} = -6x_1 + x_2 + 3x_3 + 4.$$

3.2.2. *Example.* Find the minimum possible distance between the point $P : (-3, 1)$ and the line

$$\ell : \qquad \begin{pmatrix} 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

First, since $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a direction vector of ℓ, a normal vector to ℓ is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Second, the point $(0, 3)$ is on the line (check it!), so the vector

$$\begin{pmatrix} -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

is passing through the line.

Third, by construction we have that

$$\text{proj}_{\begin{pmatrix} -2 \\ 1 \end{pmatrix}} \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \frac{\begin{pmatrix} -3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \end{pmatrix}}{\begin{pmatrix} -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \end{pmatrix}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{4}{5} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

is the shortest vector whose initial point is on the line and terminal point is P.

Hence, the minimum distance between P and ℓ is:

$$\left\| \frac{4}{5} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\| = \frac{4}{5} \left\| \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\| = \frac{4\sqrt{5}}{5}.$$

3.2.3. *Example.* Find the minimum possible distance between the point $P : (1,1,1)$ and the plane

$$\mathcal{E} : \qquad x - 2y + 3z = 4.$$

First, the normal vector to $\mathcal{E}$ is the vector $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

Second, the point $(0,-2,0)$ is on the plane (check it!), so the vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

is passing through the plane.

Third, by construction we have that

$$\operatorname{proj}_{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

is the shortest vector whose initial point is on the plane and terminal point is P.

Hence, the minimum distance between P and $\mathcal{E}$ is:

$$\left\| -\frac{1}{7} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \right\| = \frac{1}{7} \left\| \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \right\| = \frac{\sqrt{14}}{7}.$$

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