MAT 2324 – Winter 2013 – Assignment 2
Due Feb. 5 in class.

QUESTION 1. A reservoir containing water of height h empties via a small hole at the base of the reservoir. This induces a decrease in the water level equal to $k_1\sqrt{h}$, with $k_1 > 0$ a well chosen proportionality constant (we have used Torricelli's law). Suppose that this reservoir is structurally connected using a pipe to a second reservoir for which the total height of the water column above the structure from the first reservoir is H, with $H > h$ fixed. the flow from the second reservoir to the first reservoir induces an increase in the height h equal to$k_2\sqrt{H-h}$, with $k_2 > 0$ a well chosen constant. We obtain the following IVP for the height $h = h(t)$ of the first reservoir:

$$\frac{dh}{dt} = -k_1\sqrt{h} + k_2\sqrt{H-h} \qquad \text{and} \qquad h(0) = h_0,$$

where h_0 is the initial height of the reservoir. We suppose that $0 < h_0 < H$.

a) For $H > 0$, find the sole equilibrium point, denoted by $h^\star$, of the ODE. Is $0 < h^\star < H$? Is this equilibrium stable?
b) If the fill line inside the second reservoir has a height $H < 0$, the flow between the reservoirs is reversed and the ODE becomes

$$\frac{dh}{dt} = -k_1\sqrt{h} - k_2\sqrt{h-H}.$$

Is there still an equilibrium point of the system? Draw the phase diagram for valid h values. What happens in terms of existence of the solution? Make the connection with Picard's existence theorem.
c) For the situation in (b), give a lower and upper bound for the time it takes to completely empty the reservoir using respectively the maximum and minimum rates of decay for h on a well selected interval for h.
d) (Bonus) Find the exact solution when $H > 0$.

QUESTION 2. Consider the following IVP:

$$\frac{dy}{dx} = -\frac{y}{2x} + \frac{y^2}{x} \qquad \text{and} \qquad y(1) = 1.$$

a) Find the solution of the IVP and give the interval of validity of this solution.
b) Give the rectangle R on which to apply Picard's existence theorem. What happens in terms of existence of the solution when reaching the extremities of the interval of validity?

QUESTION 3. Consider the following ODE:

$$\frac{dy}{dx} = \mu y - y^3,$$

where μ is a parameter belonging to $\mathbb{R}$.

a) Identify the critical value μ_c where there is a bifurcation. Give the equilibrium points for the cases $\mu < \mu_c$ and $\mu > \mu_c$, along with the corresponding stability of the equilibria, and draw the phase line for the two cases.
b) Draw the bifurcation diagram, showing in dotted lines unstable equilibrium points and solid lines stable equilibrium points.
c) Find the general solution of this ODE and interpret it with respect to your bifurcation diagram. Think of the solution branches (the sign!) and the asymptotic limit as $x \to \infty$.

QUESTION 4. Consider a model of a certain species' population size at time t, denoted by $y(t)$, with an intrinsic per-capita growth rate, μ, along with a per-capita loss term due to predation, given by $-(y-k)^2$ (the predation may be minimal at an intermediate population size). We obtain the following differential equation:

$$\frac{dy}{dt} = y(\mu - (y-k)^2),$$

where $\mu, k \in \mathbb{R}$. Note that the case $k = 0$ is the ODE in question 3. Suppose that k is fixed and $k > 0$. Show that there exists two critical points μ_e and μ_s where bifurcation occurs. At μ_e the number of equilibria changes. At μ_s, the stability of certain equilibria change. Draw the bifurcation diagram with respect to the parameter μ. Draw phase portraits for solution for each of the following cases: $\mu < \mu_e$, $\mu_e < \mu < \mu_s$, and $\mu > \mu_s$. Explain the biological significance of each case.