Version 1

Problem 1. [5pts] Consider the following formulas of propositional logic:

$$(i)\ (p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)) \qquad (ii)\ (p \rightarrow q) \vee (p \rightarrow \neg q))$$

a) Construct the truth-tables for the two formulas.

b) For each of the above formulas, indicate whether they are a tautology, a contradiction or neither.

c) Are the two formulas logically equivalent? If so, explain why. If not, give a counterexample.

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

$\Rightarrow$ tautology

p	q	$p \rightarrow q$	$p \rightarrow \neg q$	$(p \rightarrow q) \vee (p \rightarrow \neg q)$
T	T	T	F	T
T	F	F	T	T
F	T	T	T	T
F	F	T	T	T

$\Rightarrow$ tautology

Both are tautologies, so they are equivalent.

Problem 2. [5pts] An inhabitant B of the island of knights and knaves has been accused of a crime. His lawyer, A, is also an inhabitant of the island. During the trial, the following statements are made:

A: My client is a knight only if he is innocent.

B: My lawyer is a knight if and only if I am guilty.

Can you determine the nature of A and B? Can you determine if **B** is guilty?

Can A be a knave? If so, his statement is false, so "B is a knight" = T, and "B is innocent" = F. Thus B's statement is true, i.e. "A is a knight" and "B is guilty" have the same truth-value. Thus we have a contradiction, hence <u>A is a knight</u>.

Can B be a knight? Then again "A is a knight" and "B is guilty" have the same truth-value. But from A's statement it follows that B is innocent. Contradiction, so <u>B is a knave</u>.

Finally, since B is a knave, "A is a knight" and "B is guilty" have different truth-values. Thus "B is guilty" = F, i.e. B is <u>innocent</u>.

Problem 3. [4pts] Translate the following argument into propositional logic. Use the following atoms:

p - The referees are corrupt

q - The Raptors will win the game

r - The Raptors have good players

s - The Nets bribe the referees

The Raptors have good players but if the referees are corrupt, they will not win the game. The Nets only bribe the referees if these are corrupt. If the Raptors don't win, it is because the Nets bribe the referees. Also, if the Nets don't bribe the game then the fact that the Raptors have good players is a sufficient condition for them to win. Therefore, if the Raptors don't win, the referees are corrupt.

(Note: you don't have to show that the argument is valid.)

$$\begin{array}{l} r \wedge (p \rightarrow \neg q) \\ s \rightarrow p \\ \neg q \rightarrow s \\ \neg s \rightarrow (r \rightarrow q) \\ \hline \neg q \rightarrow p \end{array}$$

Problem 4. **[4pts]** Consider the following truth table.

p	q	r	X	
T	T	T	T	✓
T	T	F	T	✓
T	F	T	T	✓
T	F	F	T	✓
F	T	T	F	✗
F	T	F	F	✗
F	F	T	F	✗
F	F	F	T	✓

Indicate for each of the following formulas whether they have the same truth table as X or not. You don't have to justify your answer.

a) $p \rightarrow (q \vee r)$

b) $p \wedge (\neg p \vee \neg q \vee \neg r)$

c) $p \vee (\neg p \wedge \neg q \wedge \neg r)$ ✓

d) $p \vee (\neg q \wedge \neg r)$ ✓

e) $(p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r)$ ✓

Problem 5. [3pts]

What is the correct translation of the following sentence into predicate logic? (As a domain of discourse we use the natural numbers.) You do not have to justify your answer.

Not every natural number which is odd is also prime, but if a natural number is not odd then is not a prime unless it is equal to 2.

a) $\neg\forall x.(Odd(x) \rightarrow Prime(x)) \wedge \forall x(\neg Odd(x) \rightarrow (Prime(x) \rightarrow x = 2))$ ✓

b) $\neg\forall x.(Odd(x) \wedge Prime(x)) \wedge \forall x \neg(Odd(x) \rightarrow (\neg Prime(x) \rightarrow x = 2))$

c) $\neg\forall x.(Odd(x) \rightarrow Prime(x)) \wedge \forall x(\neg Odd(x) \rightarrow (Prime(x) \vee x = 2))$

d) $\neg\exists x.(Odd(x) \wedge Prime(x)) \wedge \forall x(\neg Odd(x) \rightarrow (\neg Prime(x) \vee x = 2))$

e) $\neg\forall x.(Odd(x) \wedge Prime(x)) \wedge \forall x \neg(Odd(x) \rightarrow (Prime(x) \vee x = 2))$

f) $\neg\exists x.(Odd(x) \rightarrow Prime(x)) \wedge \neg\forall x.(Odd(x) \rightarrow (Prime(x) \vee x = 2))$

g) None of the above.

Problem 6. **[5pts]** For each of the following statements, indicate whether they are true or false. Note: $\mathcal{P}(X)$ denotes the powerset of X, and $|X|$ denotes the cardinality of X. You don't have to justify your answer.

a) $\emptyset \in X$ ✗

b) $x \in \{x\}$ ✓

c) $x \subseteq \{x\}$ ✗

d) $\{x\} \subseteq \{x\}$ ✓

e) $\{x\} \in \{x\}$ ✗

f) $x \in \mathcal{P}(\{x\})$ ✗

g) $\{x\} \in \mathcal{P}(\{x\})$ ✓

h) $|\mathcal{P}(\{x, y\})| = 2$ ✗

i) $|\{x, y, \{x\}, \{y\}\}| = 2$ ✗

j) $|\{x, \{x, y, z\}\}| = 2$ ✓

Problem 7. [4pts] Prove or disprove: $A - (B \cup C) = (A - B) \cap (A - C))$

Proof

$A - (B \cup C) =$

$\{x \mid x \in A \wedge x \notin B \cup C\} =$

$\{x \mid x \in A \wedge \neg (x \in B \vee x \in C)\} =$

$\{x \mid x \in A \wedge \neg x \in B \wedge \neg x \in C\} =$

$\{x \mid (x \in A \wedge \neg x \in B) \wedge (x \in A \wedge \neg x \in C)\} =$

$\{x \mid x \in A - B \wedge x \in A - C\} =$

$(A - B) \cap (A - C)$.

Problem 8. **[3pts]** Are the following predicate logical formulas are equivalent or not? If they are equivalent, demonstrate why. If they are not equivalent, give a counterexample.

$$\exists x.(\neg P(x) \vee Q(x)) \qquad \neg\forall x.(P(x) \rightarrow \neg Q(x))$$

Let $U = \mathbb{N}$, $P(x)$ ~ x is even
$Q(x)$ ~ x is odd.

Since $P(x) \rightarrow \neg Q(x)$ is true for all x,
$\neg \forall x (P(x) \rightarrow \neg Q(x))$ is false.

But $Q(3)$ is true so $\neg P(3) \vee Q(3)$ is true,
so $\exists x (\neg P(x) \vee Q(x))$ is true.

So the formulas are <u>not</u> equivalent.

Problem 9. [3pts] For each of the following arguments, determine if they are valid or not. You don't have to justify your answer.

a)
$$\begin{array}{c} p \vee q \\ \neg(r \to q) \\ \neg r \\ \hline p \end{array}$$

Valid: If $\neg(r \to q) = T$ then $r = T$, $q = F$. But if also $p \vee q = T$ then $p = T$.

b)
$$\begin{array}{c} p \wedge \neg q \\ q \to r \\ r \to p \\ \hline r \end{array}$$

Not valid: Let $p = T$, $q = F$, $r = F$. Then premises are true but conclusion false.

c)
$$\begin{array}{c} p \to \neg p \\ q \to p \\ \hline \neg q \end{array}$$

Valid: If $p \to \neg p = T$ then $p = F$. Hence also $q = F$, i.e. $\neg q = T$.