ECON 2400D, Fall 2011

Mathematical Methods of Economics

Instructor: Xiaonan Li

Assignment 1

Notes:
1. Answer **all ten questions**.
2. Completed assignment must be handed in to the economics reception at C-870 Loeb **before 11:00 am** on **OCT 13TH, 2011**.
3. Strictly **no extensions** will be allowed.
4. Clearly explain all your answers.
5. Write legibly or you may lose marks unnecessarily.

QUESTION I (15')

Given $\log_{10} 3 = 0.4771$ and $\log_{10} 16 = 1.2041$, find, without using tables or calculators,

$\log_{10} 108,\ \log_{10}(256/9),\ \log_{10} 6.$ (5' each)

Answer:

$$\log_{10} 108 = \log_{10}(27 \times 4) = \log_{10} 3^3 + \log_{10} 16^{1/2} = 3\log_{10} 3 + \frac{1}{2}\log_{10} 16$$

$$= 3 \times 0.4771 + \frac{1}{2} \times 1.2041 = 2.0334$$

$$\log_{10}(256/9) = \log_{10} 256 - \log_{10} 9 = \log_{10} 16^2 - \log_{10} 3^2 = 2 \times (\log_{10} 16 - \log_{10} 3)$$
$$= 2 \times (1.2041 - 0.4771) = 1.454$$

$$\log_{10} 6 = \log_{10} 36^{\frac{1}{2}} = \frac{1}{2}\log_{10} 36 = \frac{1}{2}\log_{10} 3^2 16^{\frac{1}{2}} = \log_{10} 3 + \frac{1}{4}\log_{10} 16 = 0.4771 + \frac{1}{4} \times 1.2041 = 0.7781$$

QUESTION II (15')

Solve the following limits (5' each)

a. $\lim\limits_{n\to\infty}(\sqrt[n]{1}+\sqrt[n]{2}+\sqrt[n]{3}+.....+\sqrt[n]{100})$

$=\lim\limits_{n\to\infty}(1+2^{\frac{1}{n}}+3^{\frac{1}{n}}+.....+100^{\frac{1}{n}})$

as $n\to\infty, \frac{1}{n}\to 0, 2^{\frac{1}{n}}, 3^{\frac{1}{n}}, ...100^{\frac{1}{n}}$ all $\to 1$

$\therefore \lim\limits_{n\to\infty}(1+2^{\frac{1}{n}}+3^{\frac{1}{n}}+.....+100^{\frac{1}{n}})=\lim\limits_{n\to\infty}(\underbrace{1+1+1+......+1}_{100})$

$=100$

b. $\lim\limits_{x\to+\infty}\frac{\ln(1+e^{4x})}{\ln(1+e^{5x})}$

$=\lim\limits_{x\to+\infty}\frac{\ln(e^{4x}e^{-4x}+e^{4x})}{\ln(e^{5x}e^{-5x}+e^{5x})}=\lim\limits_{x\to+\infty}\frac{\ln e^{4x}(e^{-4x}+1)}{\ln e^{5x}(e^{-5x}+1)}=\lim\limits_{x\to+\infty}\frac{4x+\ln(e^{-4x}+1)}{5x+\ln(e^{-5x}+1)}$

$=\frac{\lim\limits_{x\to+\infty}4x+\lim\limits_{x\to+\infty}\ln(e^{-4x}+1)}{\lim\limits_{x\to+\infty}5x+\lim\limits_{x\to+\infty}\ln(e^{-5x}+1)}=\frac{\lim\limits_{x\to+\infty}4x+\ln 1}{\lim\limits_{x\to+\infty}5x+\ln 1}=\frac{4}{5}$

c. $\lim\limits_{x\to 0}(1+2x)^{x+\frac{1}{x}}$

$=\lim\limits_{x\to 0}(1+2x)^{x}\lim\limits_{x\to 0}(1+2x)^{\frac{1}{x}}=1\times\lim\limits_{x\to 0}(1+2x)^{\frac{1}{2x}\times 2}$

$=[\lim\limits_{x\to 0}(1+2x)^{\frac{1}{2x}}]^{2}=e^{2}$

QUESTION III (4')

An Organization is doing a survey on preferences for fruits – orange, banana and apple. Totally, there are 100 people participating in it. 20 people only show their interests in orange, 15 people only in banana and 20 people only in apple. 10 people like all the three fruits, therefore, how many people choose **AT LEAST** two of the fruits? If 10 people only like orange and banana, and 13 solely prefer apple and orange, hence how many people only want both apple and banana?

Answer:
Choosing at least two kinds of fruits means that including 2 or 3 fruits. In the whole group if we subtract those who only have preference on one fruit, the remained people would at least like two fruits. So the answer is (100-20-15-20), i.e., 45.

The 45 people who at least like two fruits include those who prefer all the three fruits, and who pick up any two of the fruits, hence the answer to the number of people who want both apple and banana is (45-10-10-13), i.e., 12.

QUESTION IV (6')

James starts to save $10,000 annually since this year, supposing the annual interest rate is 5%, how much money will he have at the end of year 4? (1) Annually (2) Semi-annually (3) Quarterly compounded (2' each)

Answer:

$$\text{(1) Annually:}\quad \sum_{i=1}^{4} 10000(1+0.05)^i = 10000\frac{(1.05)^4-1}{1-\frac{1}{1.05}} = \$45256.31$$

$$\text{(2) Semi-annually:}\quad \sum_{i=1}^{4} 10000(1+0.025)^{2i} = 10000\frac{(1.025)^8-1}{1-(\frac{1}{1.025})^2} = \$45325.34$$

$$\text{(3) Quarterly:}\quad \sum_{i=1}^{4} 10000(1+0.0125)^{4i} = 10000\frac{(1.0125)^{16}-1}{1-(\frac{1}{1.0125})^4} = \$45360.76$$

QUESTION V (5')

The annual growth rate of nominal GDP of country A is 8%, and the nominal GDP is $10 Billion at year 0, so what's the value of year 4? If the annual inflation rate is 2%, what is the real GDP of year 4?

Answer:

$$GDP_4 = GDP_0(1+g)^4 = 10(1+0.08)^4 = \$13.6\text{ Billion}$$

$$RGDP_4 = \frac{GDP_0(1+g)^4}{(1+\pi)^4} = \frac{10(1+0.08)^4}{(1+0.02)^4} = \$12.57\text{ Billion OR}$$

The growth rate of real GDP is approximated as growth rate of nominal GDP less inflation rate, which is (8-2)%=6%, then you could find it as follows:

$$RGDP_4 = GDP_0(1+Rg)^4 = 10(1+0.06)^4 = \$12.62\text{ Billion. Both results of real GDP are very close.}$$

QUESTION VI (20')

The matrices **A** and **B** are given as follows:

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}, \text{ solve the questions below:}$$

(1) Find the determinants and ranks of **A** and **B**, respectively (8')

(2) $2A+3BA\,(4')$

(3) $(AB)^T(B)^{-1}\,(8')$

Answer:

(1) Apply the formula directly to get:

$|A| = 5*9*(-1)-(-1)*1*3 = -42,$

$|B| = 4*2*3+1*1*1-1*2*2-5*1*3 = 6$

Transform the matrices as:

$$A=\begin{bmatrix}1 & 5 & 1\\ 0 & 3 & 9\\ -1 & 0 & 0\end{bmatrix} \Rightarrow \text{3rd row}+\text{1st row} \Rightarrow \begin{bmatrix}0 & 5 & 1\\ 0 & 3 & 9\\ -1 & 0 & 0\end{bmatrix} \Rightarrow \text{(-3)}\times\text{2nd column}+\text{3rd column}$$

$$\Rightarrow \begin{bmatrix}0 & 5 & -14\\ 0 & 3 & 0\\ -1 & 0 & 0\end{bmatrix} \Rightarrow 5/14\times\text{3rd column}+\text{2nd column} \Rightarrow \begin{bmatrix}0 & 0 & -14\\ 0 & 3 & 0\\ -1 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix} \text{is the simplest transformation with 3 independent rows/columns, thus, the rank of A is 3.}$$

$$B=\begin{bmatrix}4 & 1 & 2\\ 5 & 2 & 1\\ 1 & 0 & 3\end{bmatrix} \Rightarrow \text{(-4)}\times\text{1st row}+\text{1st row \& (-5)}\times\text{1st row}+\text{2nd row}$$

$$\Rightarrow \begin{bmatrix}0 & 1 & -10\\ 0 & 2 & -14\\ 1 & 0 & 3\end{bmatrix} 7\times\text{2nd column}+\text{3rd column \& (-3)}\times\text{1st column}+\text{3rd column}$$

$$\Rightarrow \begin{bmatrix}0 & 1 & -3\\ 0 & 2 & 0\\ 1 & 0 & 0\end{bmatrix} 1/3\times\text{3rd column}+\text{2nd column} \Rightarrow \begin{bmatrix}0 & 0 & -3\\ 0 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix} \text{is the simplest transformation with 3 independent rows/columns, thus, the rank of B is 3.}$$

$$\text{(2) } BA=\begin{bmatrix}4 & 1 & 2\\ 5 & 2 & 1\\ 1 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 5 & 1\\ 0 & 3 & 9\\ -1 & 0 & 0\end{bmatrix}=\begin{bmatrix}4*1-2*1 & 4*5+1*3 & 4*1+1*9\\ 5*1-1*1 & 5*5+2*3 & 5*1+2*9\\ 1*1-3*1 & 1*5 & 1*1\end{bmatrix}=\begin{bmatrix}2 & 23 & 13\\ 4 & 31 & 23\\ -2 & 5 & 1\end{bmatrix},$$

$$2A+3BA=2\begin{bmatrix}1 & 5 & 1\\ 0 & 3 & 9\\ -1 & 0 & 0\end{bmatrix}+3\begin{bmatrix}2 & 23 & 13\\ 4 & 31 & 23\\ -2 & 5 & 1\end{bmatrix}=\begin{bmatrix}2+6 & 10+69 & 2+39\\ 0+12 & 6+93 & 18+69\\ -2-6 & 0+15 & 0+3\end{bmatrix}=\begin{bmatrix}8 & 79 & 41\\ 12 & 99 & 87\\ -8 & 15 & 3\end{bmatrix}$$

(3) We first need to find B^{-1}, one can either apply the adjoint matrix method or the identity matrix method. The answer follows the adjoint matrix method.

$$|B| = 4*2*3+1*1*1-2*2*1-5*1*3 = 6,$$

$$\operatorname{adj} B = \begin{bmatrix} |M_{11}| & -|M_{12}| & |M_{13}| \\ -|M_{12}| & |M_{22}| & -|M_{23}| \\ |M_{31}| & -|M_{32}| & |M_{33}| \end{bmatrix}^T = \begin{bmatrix} 2*3 & -3*5+1*1 & -2*1 \\ -1*3 & 4*3-1*2 & 1*1 \\ 1*1-2*2 & -1*4+2*5 & 4*2-5*1 \end{bmatrix}^T = \begin{bmatrix} 6 & -3 & -3 \\ -14 & 10 & 6 \\ -2 & 1 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{\operatorname{adj} B}{|B|} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{7}{3} & \frac{5}{3} & 1 \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

$$(AB)^T (B)^{-1} = \begin{bmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}^T \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{7}{3} & \frac{5}{3} & 1 \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 & 5 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 5 & 3 & 0 \\ 1 & 9 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{7}{3} & \frac{5}{3} & 1 \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4*1+5*5+1*1 & 5*3+1*9 & -4*1 \\ 1*1+2*5 & 2*3 & -1*1 \\ 2*1+1*5+3*1 & 1*3+3*9 & -2*1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{7}{3} & \frac{5}{3} & 1 \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 30 & 24 & -4 \\ 11 & 6 & -1 \\ 10 & 30 & -2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{7}{3} & \frac{5}{3} & 1 \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 30*1-24*\frac{7}{3}+4*\frac{1}{3} & -\frac{1}{2}*30+24*\frac{5}{3}-4*\frac{1}{6} & -\frac{1}{2}*30+24*1-4*\frac{1}{2} \\ 11*1-\frac{7}{3}*6+1*\frac{1}{3} & -11*\frac{1}{2}+6*\frac{5}{3}-1*\frac{1}{6} & -11*\frac{1}{2}+6*1-1*\frac{1}{2} \\ 10*1-30*\frac{7}{3}+2*\frac{1}{3} & -10*\frac{1}{2}+30*\frac{5}{3}-2*\frac{1}{6} & -10*\frac{1}{2}+30*1-2*\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{74}{3} & \frac{73}{3} & 7 \\ -\frac{8}{3} & \frac{13}{3} & 0 \\ -\frac{178}{3} & \frac{134}{3} & 24 \end{bmatrix}$$

QUESTION VII (5')

Solve the following equation system,

$$\begin{aligned} 3x_1 + x_2 &= b_1 \\ x_1 - x_2 + 2x_3 &= b_2 \\ 2x_1 + 3x_2 - x_3 &= b_3 \end{aligned}$$

, where $b_1 \neq b_2 \neq b_3 \neq 0$.

Is the solution unique?

Answer:

For a non-homogeneous equation system, the non-zero determinant of the coefficient matrix implies a unique solution.

The coefficient matrix is $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$.

The determinant is $|A| = 3*1*1 + 1*2*2 - 2*3*3 + 1*1*1 = -10$.

Furthermore, given the fact $b_1 \neq b_2 \neq b_3 \neq 0$, we should get a non-trivial solution.

QUESTION VIII (10')

A national income model is specified as

$$Y = C + I + G_0$$
$$C = a + b(Y - T)$$
$$I = c + dY$$
$$T = T_0$$

(a) Solve the equilibrium *Y*, *C* and *I* by using the Cramer's rule. (7')
(b) Find the government expenditure, tax and autonomous consumption multipliers, respectively, and indicate their relations w.r.t. income. (3')

Answer:
(a) We first write the matrix representation of the system of $Y, C,$ and I as

$$A\begin{bmatrix} Y \\ C \\ I \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -b & 1 & 0 \\ -d & 0 & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \\ I \end{bmatrix} = \begin{bmatrix} G_0 \\ a - bT_0 \\ c \end{bmatrix}, A = \begin{bmatrix} 1 & -1 & -1 \\ -b & 1 & 0 \\ -d & 0 & 1 \end{bmatrix}. \text{ And } |A| = 1 - d - b.$$

We next apply the Cramer's rule to solve for $Y, C,$ and I as

$$Y^* = \frac{\begin{vmatrix} G_0 & -1 & -1 \\ a - bT_0 & 1 & 0 \\ c & 0 & 1 \end{vmatrix}}{1 - d - b} = \frac{G_0 + c + a - bT_0}{1 - d - b}$$

$$C^* = \frac{\begin{vmatrix} 1 & G_0 & -1 \\ -b & a - bT_0 & 0 \\ -d & c & 1 \end{vmatrix}}{1 - d - b} = \frac{a - bT_0 + bc - d(a - bT_0) + bG_0}{1 - d - b}$$

$$I^* = \frac{\begin{vmatrix} 1 & -1 & G_0 \\ -b & 1 & a - bT_0 \\ -d & 0 & c \end{vmatrix}}{1 - d - b} = \frac{c + d(a - bT_0) + dG_0 - bc}{1 - d - b}$$

(b) The government expenditure, tax and autonomous consumption multipliers are

$$\frac{dY^*}{dG_0} = \frac{1}{1-d-b}$$

$$\frac{dY^*}{dT_0} = \frac{-b}{1-d-b}$$

$$\frac{dY^*}{da} = \frac{1}{1-d-b}$$

QUESTION IX (10')

Find all the partial derivatives and total differentials of the following functions:

(a) $U = -5x^3 - 12xy - 6y^5$ (3')

(b) $U = \ln xy^2 + e^{x+y} - 7x^2y^3$ (4')

(c) $U = 2x^2y - xy\ln z - 3y^2e^{xz}$ (4')

Answer:

(a) Partial derivatives

$$\frac{\partial \mathrm{U}}{\partial \mathrm{x}} = -15x^2 - 12y$$

$$\frac{\partial \mathrm{U}}{\partial \mathrm{y}} = -12x - 3y^4$$

Total differentials

$$d\mathrm{U} = \frac{\partial \mathrm{U}}{\partial \mathrm{x}}dx + \frac{\partial \mathrm{U}}{\partial \mathrm{y}}dy = -(15x^2 + 12y)dx - (12x + 3y^4)dy$$

(b) Partial derivatives

$$\frac{\partial \mathrm{U}}{\partial \mathrm{x}} = \frac{1}{xy^2}y^2 + e^{x+y} - 14xy^3$$

$$\frac{\partial \mathrm{U}}{\partial \mathrm{y}} = \frac{1}{xy^2}2xy + e^{x+y} - 21x^2y^2$$

Total differentials

$$d\mathrm{U} = \frac{\partial \mathrm{U}}{\partial \mathrm{x}}dx + \frac{\partial \mathrm{U}}{\partial \mathrm{y}}dy = (\frac{1}{xy^2}y^2 + e^{x+y} - 14xy^3)dx + (\frac{1}{xy^2}2xy + e^{x+y} - 21x^2y^2)dy$$

(c) Partial derivatives

$$\frac{\partial \mathrm{U}}{\partial \mathrm{x}} = 4xy - y\ln z - 3zy^2e^{xz}$$

$$\frac{\partial \mathrm{U}}{\partial \mathrm{y}} = 2x^2 - x\ln z - 6ye^{xz}$$

$$\frac{\partial \mathrm{U}}{\partial \mathrm{z}} = -\frac{xy}{z} - 3xy^2e^{xz}$$

Total differentials

$$d\mathrm{U} = \frac{\partial \mathrm{U}}{\partial \mathrm{x}}dx + \frac{\partial \mathrm{U}}{\partial \mathrm{y}}dy + \frac{\partial \mathrm{U}}{\partial \mathrm{z}}dz$$

$$= (4xy - y\ln z - 3zy^2e^{xz})dx + (2x^2 - x\ln z - 6ye^{xz})dy - (\frac{xy}{z} + 3xy^2e^{xz})dz$$

QUESTION X (10')

Consider a market model as follows:

$$Q_{S1} = P_1^2 + 2P_1 - 2$$
$$Q_{D1} = 4 - 2P_1 + P_2$$
$$Q_{S2} = P_2 - 1$$
$$Q_{D2} = 1 + P_1 - 2P_2$$

Solve the equilibrium levels of Q and P for each product.

Answer:

At equilibrium, $Q_{S1} = Q_{D1}$ and $Q_{S2} = Q_{D2}$, therefore,

$P_1^2 + 2P_1 - 2 = 4 - 2P_1 + P_2$ and $P_2 - 1 = 1 + P_1 - 2P_2$, rearranging to get an equation system as

$$\begin{cases} P_1^2 + 4P_1 - P_2 = 6 \\ P_1 - 3P_2 = -2 \end{cases}.$$

Substituting $P_2 = \frac{2 + P_1}{3}$ into the first equation to get

$$P_1^2 + 4P_1 - \frac{2 + P_1}{3} = 6 \Rightarrow 3P_1^2 + 11P_1 - 20 = 0 \Rightarrow (3P_1 - 4)(P_1 + 5) = 0$$

Thus, $P_1^* = \frac{4}{3}$ because price cannot be negative.

And $P_2^* = \frac{2 + P_1}{3} = \frac{10}{9}, Q_1^* = 4 - 2P^*{}_1 + P_2^* = 4 - 2 * \frac{4}{3} + \frac{10}{9} = \frac{22}{9}, Q_2^* = P_2^* - 1 = \frac{10}{9} - 1 = \frac{1}{9}.$