

1. Which of the following subsets of $F(\mathbf{R})$ are subspaces of $F(\mathbf{R})$?

$$S = \{f \in F(\mathbf{R}) \mid f(-x) + f(x) = 0\} \quad \checkmark$$

$$T = \{f \in F(\mathbf{R}) \mid f(-1)f(0) = 0\} \quad \times$$

$$U = \{f \in F(\mathbf{R}) \mid f(0) \geq 0\} \quad \times$$

$$V = \{f \in F(\mathbf{R}) \mid f(0) + f(1) = 0\} \quad \checkmark$$

(A) V and S

B. U and S

C. S and T

E. V and T

E. U and T

F. U and V .

Note:

• If $f(x) = x-1$ & $g(x) = x$, then $f, g \in T$
 but $-1 = f-g \notin T$ $\therefore T$ is not a s.s. of $F(\mathbf{R})$

• U is not closed under multⁿ by -1 :

eg. If $h(x) = 1$, then $h \in U$ but
 $-h(x) = -1$ shows $-h \notin U$.

Hence, by elimination, (A) is correct.

2. If a, b and c are scalars and u, v and w are vectors in some vector space V , which of the following statements are always true?

I. The set $\{u, v, w\}$ of vectors is linearly independent if $au + bv + cw = 0$ when $a = b = c = 0$.

This is true $\forall \{u, v, w\} \subset V$

✓ II. The set $\{u, v, w\}$ of vectors is linearly independent if $au + bv + cw = 0$ only if $a = b = c = 0$.

This is the defⁿ.

III. The set $\{u, v\}$ spans V if $\{u, v\}$ is linearly independent. No: eg $\{(1,0,0), (0,1,0)\}$ is

l.i. in \mathbb{R}^3 but doesn't span \mathbb{R}^3

✓ IV. The set $\{u, v, w\}$ spans V if every vector in V is a linear combination of $u - 2v, u + v + w$ and $v - 3w$.

$u_1 \quad v_1 \quad w_1$

A. Only I & II are true.

(B) Only II & IV are true.

C. Only II & III are true.

D. Only I & III & IV are true.

E. Only II & III & IV are true.

F. Only I is true.

Clearly,

$$V = \text{span}\{u_1, v_1, w_1\} \subseteq \text{span}\{u, v, w\},$$

Since each of u_1, v_1 and w_1 is

a l.c. of $\{u, v, w\}$

Thus $\{u, v, w\}$ spans V .

3. Let $A = \begin{bmatrix} -1 & 2 & -1 \\ -5 & 7 & -3 \\ 3 & -4 & 2 \end{bmatrix}$. Which one of the following is true?

A. The second column of A^{-1} is $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$.

B. The third column of A^{-1} is $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

C. The third row of A^{-1} is $[-1 \ 2 \ 3]$.

D. The first row of A^{-1} is $[-2 \ 0 \ 1]$.

E. The matrix A is not invertible.

F. The second row of A^{-1} is $[-1 \ -1 \ 2]$.

$$\left[\begin{array}{ccc|ccc} -1 & 2 & -1 & 1 & 0 & 0 \\ -5 & 7 & -3 & 0 & 1 & 0 \\ 3 & -4 & 2 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & -3 & 2 & -5 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 1 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 3 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 2 & 3 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 2 & 3 \end{array} \right]$$

STOP!

4. Suppose $n \geq 2$. In a linear system $Ax = b$, with n equations and n unknowns, the rank of A is $n-1$ and the rank of the augmented matrix $[A \mid b]$ is also $n-1$. Which one of the following statements is true? ||29

A. The system has no solution.

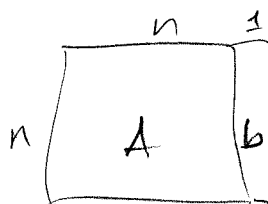
B. The system has a unique solution.

C. The system has infinitely many solutions.

D. The system has exactly $n-1$ solutions.

E. The determinant of A is non-zero.

F. Such a system cannot exist.



$$\# \text{ rank } A = \text{rank}(A|b) = n-1 < n$$

= # variables

Thus the system is consistent
with ∞ many solns

5. Compute $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^{2012}$

$$\left[\begin{array}{c|c} I_2 & P \\ \hline 0 & N \end{array} \right] \left[\begin{array}{c|c} I_2 & P \\ \hline 0 & N \end{array} \right] = \left[\begin{array}{c|c} I_2 & P+PN \\ \hline 0 & N^2 \end{array} \right]$$

A. $\begin{bmatrix} 1 & 0 & 2012 & 0 \\ 0 & 1 & 0 & 0 \\ 2012 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 2012 & 2012 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2012 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

F. $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$N^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad PN = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = N$$

$$A^3 = \begin{bmatrix} 1 & P+PN \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & P \\ 0 & N \end{bmatrix} = \begin{bmatrix} 1 & P+PN+PN^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & P+PN \\ 0 & 0 \end{bmatrix} = A^2$$

$$\therefore A^{2012} = A^2 = \begin{bmatrix} I_2 & P+PN \\ 0 & 0 \end{bmatrix}$$

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6. Let M_{22} denote, as usual, the vector space of real 2×2 matrices and let $K = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, and define

$$C = \{A \in M_{22} \mid KA = AK\}.$$

A. C is a subspace of M_{22} of dimension 1

B. C is a subspace of M_{22} of dimension 2

C. C is a subspace of M_{22} of dimension 3

D. C is a subspace of M_{22} of dimension 4

E. C is a subspace of M_{22} of dimension 0

F. C is not a subspace of M_{22} .

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} b & 2a \\ d & 2c \end{bmatrix} = \begin{bmatrix} 2c & 2d \\ a & b \end{bmatrix}$$

$$\therefore b = 2c, \quad 2a = 2d$$

$$\therefore \begin{matrix} b=c \\ a=d \end{matrix} \quad \therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & 2c \\ c & d \end{bmatrix} = dI_2 + cK;$$

Moreover, $\{dI_2, K\}$ is clearly l.o.s. $\therefore \dim C = 2$

7. Let A be a 6×4 matrix. Answer the following questions:

(1) Can the system $Ax = 0$ have a non trivial solution?

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it will always have ∞ many

(2) Can the columns of A span \mathbf{R}^6 ?

No, we need at least 6 vectors to span \mathbf{R}^6

(3) Can the columns of A be linearly independent?

A. Yes, No, Yes.

B. Yes, Yes, Yes.

C. Yes, No, No.

D. No, No, Yes.

E. No, No, No.

F. No, Yes, Yes.

Yes, when $\text{rank } A = 4$, which is possible.

\therefore Yes, No, Yes $(A = \begin{bmatrix} I_4 \\ 0-0 \\ 0-0 \end{bmatrix})$

8. The set of vectors $\{(3, 0, 4), (0, 1, 0), (-4, 0, 3)\}$ is an orthogonal basis of \mathbf{R}^3 . Find $c_3 \in \mathbf{R}$ such that

$$(0, 1, 1) = c_1(3, 0, 4) + c_2(0, 1, 0) + c_3(-4, 0, 3) \quad \text{for some scalars } c_1, c_2 \in \mathbf{R}.$$

That is, find the third Fourier coefficient of $(0, 1, 1)$ with respect to the ordered orthogonal basis of \mathbf{R}^3 above.

A. $\frac{3}{25}$

B. $\frac{3}{5}$

C. 3

D. 1

E. $\frac{1}{25}$

F. $-\frac{3}{25}$

$$\begin{aligned} \text{We know } c_3 &= \frac{(0, 1, 1) \cdot (-4, 0, 3)}{(-4)^2 + 0^2 + 3^2} \\ &= \frac{3}{25} \end{aligned}$$

9. Which two of the following statements are **false**?

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- (i) For all invertible $n \times n$ matrices A and B , $\det(A^{-1}BA) = \det B$
 (ii) For all invertible $n \times n$ matrices A and B , $\det(A^{-1}B^{-1}AB) = 1$
 (iii) For all $n \times n$ matrices A and B , $\underline{\underline{(\underline{A^t B^t})^t = AB}}$
 (iv) For all invertible $n \times n$ matrices A and B , $\underline{\underline{(ABA^{-1})^{-1} = A^{-1}B^{-1}A}}$
 (v) For all $n \times n$ matrices A and B , $\det(A^t B) = \det(B^t A)$

True

True

$$\textcircled{AB^t}^t = BA!$$

$$\textcircled{ABA^{-1}}^{-1} = AB^{-1}A^{-1}$$

- A. (i) and (iii)
 B. (ii) and (iii)
 C. (iii) and (iv)
 D. (ii) and (iv)
 E. (ii) and (v)
 F. (i) and (v)

It's possible that $AB \neq BA$,
 and $A^{-1}B^{-1}A \neq AB^{-1}A^{-1}$.

10. Let A be an $n \times n$ matrix. One of the the following statements in not equivalent to the statement:

The number 0 is not an eigenvalue of A .

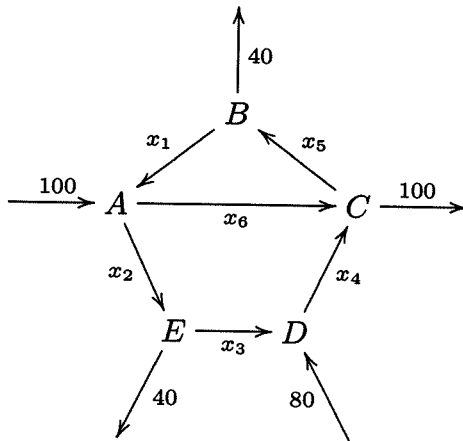
Which one?

$$\text{So } 0 \neq \det(A - 0I) \\ = \det A$$

$\therefore A$ is invertible

- A. The homogeneous system $Ax = 0$ has a non-trivial solution.
 B. A is invertible. ✓
 C. The determinant of A is not zero. ✓
 D. The columns of A span \mathbf{R}^n . ✓
 E. The rows of A are linearly independent. ✓
 F. The rank of A is n . ✓

11. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the the traffic flow, **together with all the constraints** on the variables $x_i, i = 1, \dots, 6$.

(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.)

$2 = 1(\text{eqns}) + \frac{1}{2} + \frac{1}{2}$
if "=" missing

Intersection Flow IN = Flow OUT constraints

A $100 + x_1 = x_2 + x_6$

B $x_5 = x_1 + 40$

C $x_4 + x_6 = 100 + x_5$

D $x_3 + 80 = x_4$

E $x_2 = x_3 + 40$

$\frac{1}{2}$ - missed on intersection

and $x_i \geq 0$ (one-way streets)

$x_i \in \mathbb{Z}$ (whole numbers of cars.)

O'if "A = $x_1 - x_2 - x_6 + 100$ " etc

11(b). The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccc|cc|c} 1 & 0 & 0 & 0 & \Delta & t & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 60 \\ 0 & 0 & 1 & 0 & -1 & 1 & 20 \\ 0 & 0 & 0 & 1 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

2 ; If error
1 25 53
0 else

$$x_1 = -40 + \Delta$$

$$x_2 = 60 + \Delta - t$$

$$x_3 = 20 + \Delta - t$$

$$x_4 = 100 + \Delta - t$$

; $\Delta, t \in \mathbb{R}$

$$x_5 = \Delta$$

$$x_6 = t$$

only 1/2 if these
are missing

c) If \overline{AC} were closed due to roadwork, find the minimum flow along \overline{ED} , using your results from (b).

(You must justify all your answers.)

Since $x_6 =$ flow along \overline{AC} , $t=0$.

Flow along \overline{ED} is $x_3 = 20 + \Delta$ (unless min flow ≥ 0 or not given) (1/2)

Constraints (in light of $t=0$)

$$x_1 \geq 0 \Rightarrow \Delta \geq 40$$

$$x_2 = 60 + \Delta \geq 0 \checkmark$$

$$x_3 = 20 + \Delta \geq 0 \checkmark$$

$$x_4 = 100 + \Delta \geq 0 \checkmark$$

$$x_5 = \Delta \geq 0 \checkmark$$

0 - if answer is negative

1 - correct

answer
(with an attempt at justification)
1 - justified

(" ≥ 40 " 1/2 if all constraints considered and most are correct)

1 but if " faults, and in fluke 1/2

$$\therefore x_3 = 20 + \Delta \geq 60$$

(1) if all correct but made one error in equalities and got wrong answer

1/2 if correct but did not consider all constraints

12. Let $U = \text{span}\{(1, 0, 0, 1), (0, 1, 0, 0), (1, 1, 0, 0), (1, 4, 0, -1)\}$

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2 a) Find a basis of U which is a subset of the given spanning set.

3 b) Find an orthogonal basis of U .

2 c) Find the best approximation to $(1, -1, 2, -1)$ by vectors in U .

2 d) Extend your basis in (b) to an orthogonal basis of \mathbb{R}^4 .

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 4 & 0 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$. Then $\text{col } A = U$.

$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \therefore \{u_1, u_2, u_3\}$ is such a basis
1 + just

b) Since $u_1 \cdot u_2 = 0$, we set $w_1 = u_1, w_2 = u_2$ and

$$w_3 = u_3 - \frac{u_3 \cdot w_1}{\|w_1\|^2} w_1 - \frac{u_3 \cdot w_2}{\|w_2\|^2} w_2$$

2 - if used as correctly but answer wasn't orthog

$$\begin{pmatrix} 1, 0, 0, 1 \\ 0, 1, 0, 0 \\ \frac{1}{2}, 0, 0, \frac{1}{2} \\ \text{or } (1, 0, 0, -1) \end{pmatrix} = (1, 1, 0, 0) - \frac{1}{2}(1, 0, 0, 1) - (0, 1, 0, 0)$$

$$= \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right) \therefore \{w_1, w_2, w_3\} \text{ is an}$$

orthogonal basis for U . (Or $w_3 = (1, 0, 0, 1)$ will do too)

c) $\text{proj}_U (1, -1, 2, -1) = \frac{(1, -1, 2, -1) \cdot w_1}{\|w_1\|^2} w_1 + \frac{(1, -1, 2, -1) \cdot w_2}{\|w_2\|^2} w_2$

$$+ \frac{(1, -1, 2, -1) \cdot w_3}{\|w_3\|^2} w_3$$

1 - correct form $(\frac{1}{2} + \frac{1}{2})$

$$= 0 + \frac{(-1)}{1} (0, 1, 0, 0) + \frac{2}{2} (1, 0, 0, -1)$$

$$= \underline{(1, -1, 0, -1)} \quad \text{--- } 1 \left(\frac{1}{2} + \frac{1}{2}\right)$$

12(d) We know that $\begin{bmatrix} w_1 \\ w_2 \\ w_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix},$

so $\{w_1, w_2, w_3', e_3\}$ is a basis of \mathbb{R}^4 ; we apply G-S

$$\text{to obtain } w_4 = e_3 - \frac{e_3 \cdot w_1}{\|w_1\|^2} w_1 - \frac{e_3 \cdot w_2}{\|w_2\|^2} w_2 - \frac{e_3 \cdot w_3'}{\|w_3'\|^2} w_3'$$

$$= e_3 - 0 - 0 - 0$$

(or: simply note that e_3 is orthogonal to $w_1, w_2, \& w_3'$.

2 - orthog. basis + some stuff

$$13. A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

- a) Compute $\det(A - \lambda I_3)$ and hence show that the eigenvalues of A are 0 and 2.
 b) Find a basis of $E_0 = \{v \in \mathbb{R}^3 \mid Av = 0\}$.
 c) Find a basis of $E_2 = \{v \in \mathbb{R}^3 \mid Av = 2v\}$.
 d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Explain why your choice of P is invertible.

$$\begin{aligned} a) \quad \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} & \stackrel{\text{row 2}}{=} \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda) \{ (\lambda-1)^2 - 1 \} \\ & = (2-\lambda) \{ \lambda^2 - 2\lambda + 1 - 1 \} \\ & = (2-\lambda) \lambda(\lambda-2) \\ & = -(2-\lambda)^2 \lambda \end{aligned}$$

$$\therefore \det(A - \lambda I_3) = 0 \Leftrightarrow \lambda = 0 \text{ or } \lambda = 2$$

$$b) E_0 = \ker(A - 0I) = \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \{ (-1, 0, 1) \}$$

$\{ (-1, 0, 1) \}$ is a basis for E_0 .

$$c) E_2 = \ker(A - 2I) = \ker \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= \text{span} \{ (0, 1, 0), (1, 0, 1) \} \quad \therefore \{ (0, 1, 0), (1, 0, 1) \}$ is a basis for E_2 .

$$d) \text{ Set } P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \text{ Then } P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \text{ We}$$

know P is invertible because $\dim E_0 + \dim E_2 = 1 + 2 = 3$; OR :

$$\det P = \begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \stackrel{\text{row 2}}{=} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2 \neq 0.$$

14. Let $u = (1, -1, 1)$ and define a linear transformation $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$S(v) = u \times v, \quad v \in \mathbb{R}^3,$$

where " \times " denotes the cross product. (You do not have to prove that S is linear.)

(a) If $(x, y, z) \in \mathbb{R}^3$, show that $S(x, y, z) = (-y - z, x - z, x + y)$.

(b) Find the standard matrix of S .

(c) Find a basis for $\ker S$ and describe it geometrically.

(d) Find a basis for $\text{im } S$ and describe it geometrically.

$$a) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix} = (-z-y, -(z-x), y+x) = (-y-z, x-z, x+y) \text{ as required.}$$

$$b) A = [S e_1 \ S e_2 \ S e_3] = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \text{ (by (a)) is}$$

the standard matrix of S

$$c) \ker S = \ker \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \text{span} \{ (1, -1, 1) \} \therefore \{ u \} \text{ is a basis for } \ker S$$

(as we expect). Thus $\ker S$ is the line through 0 with direction $(1, -1, 1)$.

$$d) \text{im } S = \text{col } A = \text{span} \{ (0, 1, 1), (-1, 0, 1) \}; \text{ hence } \{ u_2, u_3 \}$$

is a basis of $\text{im } S$. Thus $\text{im } S$ is the plane through 0

$$\text{with normal } u_2 \times u_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = (1, -1, 1) = u. \text{ (As expected)}$$

15. State whether the following are true or false.

If you say the statement may be false, you must give an explicit example - with numbers! If you say the statement is true, you must give a clear explanation -e.g. by quoting a theorem from class.

i) Every diagonalizable matrix is invertible. $\left(\frac{1}{2} \text{ correct} + \frac{1}{2} \text{ just "}. \right)$

FALSE

eg $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonal (and so diagonalizable) but $\text{rank } A = 1 < 2$ so A is not invertible.

ii) If a 2 by 2 matrix A satisfies $A^2 = 0$, then $A = 0$.

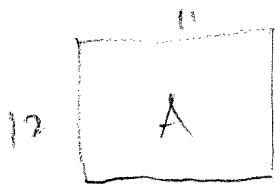
FALSE

If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $A \neq 0$

$$\text{but } A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

iii) The rows of a 12×11 matrix are always linearly dependent.

TRUE



12 vectors in \mathbb{R}^{11} are always l.o.s. since $\dim \mathbb{R}^{11} = 11 \geq \text{size of any l.o.s. set}$