Chapter 2: The Logic of Compound Statements

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Outline

- Central notion of deductive logic: argument form
- Argument: sequence of statements whose goal is to establish the truth of an assertion
- The assertion at the end of the sequence is called the conclusion while the preceding statements in an argument are called premises

The goal of an argument is to show that the truth of the conclusion follows necessarily from the truth of the premises.

Example

If x is a real number such that $x < -5$ or $x > 5$, then $x^2 > 25$. Since $x^2 \not> 25$ then $x \not< -5$ and $x \not> 5$.

We can introduce the letters p, q, and r to represent statements that occur within our argument:

> If p or q, then r.
>
> Therefore, if not r, then not p and not q.

Example

Fill in the blanks in the argument (b) so that it has the same form as the argument (a). Then, write the common form of the argument using letters to replace the individual statements.

(a)

> If it rains today or I have a lot of work to do, I won't go for a walk.
>
> I have a lot of work to do.
>
> Therefore, I won't go for a walk.

(b)

> If MTH110 is easy or ____ then ____.
>
> I will study hard.
>
> Therefore, I will get an A in this course.

Solution:

1. I will study hard.
2. I will get an A in this course.

Common form of the arguments:

> If p or q, then r.
>
> q
>
> Therefore, r.

Statements

Definition

A statement (or, proposition) is a sentence that is true or false but not both.

Examples

1. "The area of the circle of radius r is $r^2\pi$" is a statement. So is "$\sin(\pi/2) = 0$". The first is a true statement, while the second one is false.
2. "$x + 2 \geq y$" is not a statement; namely, for some values of x and y, e.g. $x = 1$, $y = 2$, it is true, while for some other values (e.g. $x = -1$, $y = 2$), it is false.

Compound Statements

- We want to build more complex logical expressions starting with simple statements.
- We will introduce three new logical symbols (connectives):

1. $\sim$ (NOT)
2. $\wedge$ (AND)
3. $\vee$ (OR)

- "$\sim p$" means "not p" or "It is not the case that p" and is called the negation of p.
- "$p \wedge q$" means "p and q" and is called the conjunction of p and q.
- "$p \vee q$" means "p or q" and is called the disjunction of p and q.

In logical expressions, the symbol $\sim$ is evaluated before $\wedge$ or $\vee$, since it binds statements in a stronger way than conjunction or disjunction. (This is similar to the fact that, in arithmetic, we evaluate multiplication before + or -)
For example, we can simplify the expression

$$(\sim p) \vee q$$

as

$$\sim p \vee q$$

- “p but q” is often translated as “p and q”.
  For example, the statement “It is not snowing but it is cold.” can be written as
  $$\sim p \wedge q$$

- “neither p nor q” means “not p and not q”.
  For instance, “It is neither snowing nor it is cold” can be written as
  $$\sim p \wedge \sim q$$

- Mathematical inequalities can be written using AND and OR.

1. $x \leq a$ means $x < a$ OR $x = a$.
2. $a \leq x \leq b$ means $a \leq x$ AND $x \leq b$.

Example

Suppose x is a particular real number. let p, q, and r represent "$0 < x$", "$x < 3$", and "$x = 3$". Write the following inequalities symbolically

(a) $x \leq 3$.

$$q \vee r$$

(b) $0 < x < 3$

$$p \wedge q$$

(c) $0 < x \leq 3$

$$p \wedge (q \vee r)$$

Truth Values

- We defined statements as assertions which are either true or false (but not both).
- If we have a compound statement built from simpler statements using $\sim$, $\wedge$, and $\vee$, how do we decide whether it is true or false?
- For that purpose, we use truth tables.

Definition

If p is a statement, then $\sim p$ is false when p is true, and $\sim p$ is true when p is false.

p	$\sim p$
T	F
F	T

Definition

The conjunction of two statements $p \wedge q$ is true only in the case when both statements p and q are true. In all other cases, $p \wedge q$ is false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition

The disjunction of two statements $p \vee q$ is false only in the case when both statements p and q are false. In all other cases, $p \vee q$ is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example

Construct the truth table for the statement

$$(p \vee q) \wedge \sim (p \wedge q).$$

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

This statement is true when exactly one of the statements p and q is true. For that reason, it is called the exclusive OR (or XOR) and is sometimes written as

$$p \oplus q$$

Example

Construct the truth table for the statement form

$$p \wedge (\sim q \vee r)$$

p	q	r	$\sim q$	$\sim q \vee r$	$p \wedge (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	T	T	F
F	F	F	T	T	F

Logical Equivalence

- Obviously, two statements such as “It is cold and it is snowing” and “It is snowing and it is cold”, even though they are different as sentences, are two different ways of stating the same thing from the logical point of view.
- This is because $p \wedge q$ and $q \wedge p$ have the same truth values for the same values of p and q.

Definition

Two statement forms P and Q are called logically equivalent if they have the same truth values for all possible truth values of statement variables $p, q, r, \ldots$
We write that as

$$P \equiv Q$$

Testing Whether Two Statement Forms P and Q Are Logically Equivalent

1. Construct a combined truth table with the last two columns being the truth values for P and Q.
2. Check each row of the table; if the values of P and Q are the same in each row, the statement forms P and Q are equivalent. If there is at least one row where the values of P and Q differ, the statement forms are not equivalent.

Example

The double negation property $\sim(\sim p) \equiv p$:

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Since p and $\sim(\sim p)$ have the same truth values, they are equivalent.

Example

Show that the statements $\sim (p \vee q)$ and $\sim p \vee \sim q$ are not logically equivalent.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	T

De Morgan's Laws

Example

Show that the statement forms $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are logically equivalent.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Therefore

$$\boxed{\sim(p \vee q) \equiv \sim p \wedge \sim q}$$

This important equivalence is called a De Morgan's Law.
One can show similarly that

$$\boxed{\sim(p \wedge q) \equiv \sim p \vee \sim q}$$

Examples

Write negations for each of the following statements:

(a) The train is late or my watch is slow.
The train is not late and my watch is not slow

(b) It is cold today and it is sunny outside.
It is not cold today or it is not sunny outside.

Example

Use De Morgan's laws to write the negation of

$$-3 \leq x < 2.$$

Solution: The given statement is equivalent to

$$-3 \leq x \text{ and } x < 2.$$

By De Morgan's laws, the negation is

$$-3 \not\leq x \text{ or } x \not< 2$$

which is equivalent to

$$-3 > x \text{ or } x \geq 2$$

Tautologies and Contradictions

Definition

A tautology is a statement form that is always true regardless of the truth values assigned to the statement variables.
A contradiction is a statement form which is always false, regardless of the truth values assigned to the statement variables.

Example

Show that the statement $p \vee \sim p$ is a tautology and that the statement $p \wedge \sim p$ is a contradiction.

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

- Notation: From now on, an arbitrary tautology will be denoted **t**, and an arbitrary contradiction will be denoted **c**

Example

If **t** is a tautology and **c** is a contradiction, show that

$$p \wedge \mathbf{t} \equiv p, \quad p \wedge \mathbf{c} \equiv \mathbf{c}$$

Solution:

p	$\mathbf{t}$	$p \wedge \mathbf{t}$	p	$\mathbf{c}$	$p \wedge \mathbf{c}$
T	T	T	T	F	F
F	T	F	F	F	F

A List of Useful Logical Equivalences

1. Commutative Laws

$$p \wedge q \equiv q \wedge p, \;\; p \vee q \equiv q \vee p$$

2. Associative Laws

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r), \;\; (p \vee q) \vee r \equiv p \vee (q \vee r)$$

3. Distributive Laws

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r), \;\; p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

4. Identity Laws

$$p \wedge \mathbf{t} \equiv p, \;\; p \vee \mathbf{c} \equiv p$$

5. Negation Laws

$$p \vee \sim p \equiv \mathbf{t}, \;\; p \wedge \sim p \equiv \mathbf{c}$$

6. Double Negation Law

$$\sim(\sim p) \equiv p$$

7. Idempotent Laws

$$p \wedge p \equiv p, \quad p \vee p \equiv p$$

8. Universal Bound Laws:

$$p \vee \mathbf{t} \equiv \mathbf{t}, \quad p \wedge \mathbf{c} \equiv \mathbf{c}$$

9. De Morgan's Laws

$$\sim(p \wedge q) \equiv \sim p \vee \sim q, \quad \sim(p \vee q) \equiv \sim p \wedge \sim q$$

10. Absorption Laws:

$$p \vee (p \wedge q) \equiv p, \quad p \wedge (p \vee q) \equiv p$$

11. Negations of **t** and **c**

$$\sim \mathbf{t} \equiv \mathbf{c}, \quad \sim \mathbf{c} \equiv \mathbf{t}$$

Using Logical Equivalences to Simplify Statement Forms

Suppose we want to show that

$$\sim(\sim p \vee q) \vee (p \wedge q) \equiv p$$

We can proceed as follows:

$$\begin{aligned}
\sim(\sim p \vee q) \vee (p \wedge q) &\equiv (\sim(\sim p) \wedge \sim q) \vee (p \wedge q) && \text{(De Morgan)} \\
&\equiv (p \wedge \sim q) \vee (p \wedge q) && \text{(Double neg.)} \\
&\equiv p \wedge (\sim q \vee q) && \text{(Distributivity)} \\
&\equiv p \wedge (q \vee \sim q) && \text{(Commutativity)} \\
&\equiv p \wedge \mathbf{t} && \text{(Negation Law)} \\
&\equiv p && \text{(Identity)}
\end{aligned}$$

Conditional Statements

- A conditional statement form is a sentence of the form

  "If p then q."

- For example, one such statement is

  "If 32 is a power of 2, then 32 is an even number."

  or:

  "If I study hard, I will get a good mark in this course."

- The notation for such a statement is

$$p \to q$$

  and p is called the hypothesis (or, the premise) and q is the conclusion of the conditional statement form.

- Question: When is a conditional statement true?
- The only way a conditional argument can be false is if we derive a false conclusion from a true hypothesis. In all other case, this conditional statement will be true.
- In a way, if the hypothesis is false to start with, the truth or falsity of the conclusion is irrelevant, so if p is false, the conditional statement is said to be true by default.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example

Construct a truth table for the statement form

$$\sim p \vee q \rightarrow \sim q$$

Solution:

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$\sim p \vee q \rightarrow \sim q$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Example

Show the logical equivalence

$$p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$$

Solution:

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$p \vee q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Representing If-Then as Or

- It can be shown that

$$\boxed{p \to q \equiv \sim p \vee q}$$

Example

Consider the statement

"Either you fix my sink or I won't pay the rent."

Let $\sim p$ stand for "I won't pay the rent." and let q denote "You fix my sink.". Then, we have the statement

$$\sim p \vee q.$$

This is equivalent to

$$p \to q$$

which translates as

"If I pay the rent, then you fix my sink."

Negation of a Conditional Statement

We prove the following logical equivalence

$$\boxed{\sim (p \to q) \equiv p \wedge \sim q}$$

$$\begin{aligned} \sim (p \to q) &\equiv \sim (\sim p \vee q) \\ &\equiv \sim (\sim p) \wedge (\sim q) \qquad \text{(De Morgan)} \\ &\equiv p \wedge \sim q \qquad \text{(Double neg.)} \end{aligned}$$

- Therefore,

  The negation of "if *p* then *q*" is equivalent to "*p* and not *q*"

Example

The negation of the statement

> “If it doesn’t rain today, I will go for a bike ride.”

is

> “It doesn’t rain today and I will not go for a bike ride.”

Contrapositive Statements

Definition

The contrapositive of a conditional statement "If p then q" is

$$\text{If } \sim q \text{ then } \sim p.$$

Symbolically, the contrapositive of $p \rightarrow q$ is

$$\sim q \rightarrow \sim p$$

Example

The contrapositive of the statement

"If I study hard, then I am doing well in the course."

is

"If I am not doing well in the course, then I don't study hard."

Fact: (proved in the homework exercises)

$$\boxed{p \to q \equiv \sim q \to \sim p}$$

In other words,

Any conditional statement is equivalent to its contrapositive statement.

Converse and Inverse Statements

Definition

Given a conditional statement “If p then q.”, then

1. The converse is “If q then p.” ($q \to p$)
2. The inverse is “If $\sim p$ then $\sim q$” ($\sim p \to \sim q$)

For example, given the statement

“If it doesn’t rain today, I can go for a bike ride.”

its converse is

“If I can go for a bike ride, it doesn’t rain today.”

and its inverse is

“If it rains today, I can’t go for a bike ride.”

Facts:

1. A conditional statement and its converse are not equivalent.
2. A conditional statement and it inverse are not equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent, since they are contrapositives of each other.

Only If and the Biconditional Statements

Definition

If p and q are statements,

p **only if** q

means

"If not q then not p."

or, equivalently,

"If p then q."

Example

Consider the statement

The Leafs will get into the play-offs only if they win the tomorrow's game.

One way to rewrite that statement as a conditional one:

If they don't win the tomorrow's game, the Leafs won't get into the play-offs.

Another way to do it:

If the Leafs get into the play-offs, they will win the tomorrow's game.

Warning: The statement "p only if q" does not mean "p if q".

Definition

Given statements p and q, the biconditional of p and q is

"p if, and only if q."

and we write it as

$$p \leftrightarrow q$$

The biconditional is true only in those cases when p and q have the same truth values.
We often write it in short as

p **iff** q

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- It can be shown (using truth tables, e.g.) that

$$\boxed{p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)}$$

- Now that we have encountered all logical connectives that will be used, we can classify them according to how "strong" they are as logical operations:

1. $\sim$
2. $\wedge$ and $\vee$
3. $\to$ and $\leftrightarrow$

Example

Rewrite the following statement as a conjunction of two if-then statements:

This integer is even if, and only if, when divided by 2 it gives the remainder 0.

We can rewrite it as follows:

If this integer is even, then it gives the remainder 0 when divided by 2 and, if this integer gives the remainder 0 when divided by 2, then it is even.

So, if p stands for “This integer is even.” and q for “This integer gives the reminder 0 when divided by 2.”, we have

$$(p \to q) \wedge (q \to p)$$

which is equivalent to

$$(\sim p \vee q) \wedge (\sim q \vee p)$$

Necessary and Sufficient Conditions

Definition

If p and q are statements,

(a) p is a sufficient condition for q means

"If p then q."

(b) p is a necessary condition for q means

"If not p then not q."

or, equivalently,

"If q then p."

So, "p is a necessary and sufficient condition for q" means

"p if, and only if, q."

Example

For example, the statement

If Mary is eligible to vote, then she is at least 18 years old.

means the following

The condition that Mary is eligible to vote is *sufficient* to ensure the condition that she is at least 18 years old.

On the other hand,

The condition that Mary is at least 18 years old is *necessary* for the condition that Mary is eligible to vote to be true.

Arguments

Consider the following sequence of statements

If the function $f(x)$ is continuous on $[a, b]$ then $\int_a^b f(x)\, dx$ exists.

The function $f(x)$ is continuous on $[a, b]$.

$\therefore$ $\int_a^b f(x)\, dx$ exists.

- The symbol $\therefore$ indicates the conclusion of a logical argument.

We can replace the statements with variables p and q to get the abstract form of the argument

$$\begin{array}{rl} & p \to q \\ & p \\ \therefore & q \end{array}$$

Our goal in this section will be to see how we can determine which abstract arguments are logically sound or not and how to prove or disprove them.

Definition

An argument is a sequence of statements. All statement forms, except the very last one, are called premises (or, assumptions, or hypotheses) while the last statement is called the conclusion.

The symbol $\therefore$ is read "*therefore*".

An argument is valid if, whenever all the premises are true, the conclusion is also true.

How To Test An Argument For Validity

1. Determine the premises and the conclusion of the argument.
2. Construct a truth table showing the truth values of all the premises as well as the conclusion.
3. If there is a row of the truth table in which all the premises are true while the conclusion is false, the argument will be invalid. If, for every row where all the premises are true, the conclusion is also true, the argument will be valid.

Example

Show that the argument

$$p \to q \vee \sim r$$

$$q \to p \wedge r$$

$$\therefore\ p \to r$$

is invalid.

p	q	r	$\sim r$	$q\vee \sim r$	$p \wedge r$	$p \to q\vee \sim r$	$q \to p \wedge r$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	F
T	F	T	F	F	T	F	T	T
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	T
F	T	F	T	T	F	T	F	T
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

Example

Show that the following argument is valid

$$p \vee (q \vee r)$$

$$\sim r$$

$$\therefore\ p \vee q$$

p	q	r	$q \vee r$	$p \vee (q \vee r)$	$\sim r$	$p \vee q$
T	T	T	T	T	F	
T	T	F	T	T	T	T
T	F	T	T	T	F	
T	F	F	F	T	T	T
F	T	T	T	T	F	
F	T	F	T	T	T	T
F	F	T	T	T	F	
F	F	F	F	F	T	

Modus Ponens and Modus Tollens

- An argument with two premises and a conclusion is called a syllogism. the first premise is called a *major* premise, while the second premise is called the *minor* premise.
- Modus Ponens is a syllogism of the form

$$\begin{array}{rl} & p \rightarrow q \\ & p \\ \therefore & q \end{array}$$

- Modus Ponens is a valid argument form:

p	q	$p \to q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

- Modus Tollens is an argument of the form

$$\begin{array}{ll} & p \to q \\ & \sim q \\ \therefore & \sim p \end{array}$$

- It can be shown (using truth tables, e.g.) that Modus Tollens is a valid argument form.

Example

An example of an argument in the Modus Tollens form is:

If my employee evaluation is good, I will get a raise.

I won't get a raise.

∴ My employee evaluation is not good.

Example

For the following premises find the conclusion which will make it valid and justify the reasoning

If this figure is a triangle, then the sum of its angles is 180°.

The sum of the angles of this figure is not 180°.

$\therefore$ ____

Solution: The conclusion which will make the argument valid is

This figure is not a triangle.

The justification is provided by Modus Tollens.

More Valid Argument Forms (Rules of Inference)

1. Generalization: this rule can have two different forms

   (a) $\begin{array}{l} p \\ \therefore p \vee q \end{array}$ (b) $\begin{array}{l} q \\ \therefore p \vee q \end{array}$

2. Specialization: again, this rule can have two forms

   (a) $\begin{array}{l} p \wedge q \\ \therefore p \end{array}$ (b) $\begin{array}{l} p \wedge q \\ \therefore q \end{array}$

3. Elimination:

(a)	$p \vee q$ $\sim q$ $\therefore p$	(b)	$p \vee q$ $\sim p$ $\therefore q$

Example

For example, suppose we know that

$$x - 5 = 1 \text{ or } x + 2 \leq 3$$

and

$$x > 2$$

Then,

$$x + 2 \nleq 3$$

$$\therefore x - 5 = 1$$

by the Elimination Rule.

4. Transitivity:

$$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \therefore & p \rightarrow r \end{array}$$

Example

If I go to the movies, I won't finish my assignment.

If I don't finish my assignment, my grade will drop.

∴ If I go to the movies, my grade will drop.

5. Proof by Division Into Cases:

$$\begin{array}{rl} & p \vee q \\ & p \rightarrow r \\ & q \rightarrow r \\ \therefore & r \end{array}$$

Example

If I get a Christmas bonus, I will buy a computer.

If I sell my car, I will buy a computer.

I will either get a Christmas bonus or sell my car.

$\therefore$ I will buy a computer.

Example

You are about to leave for classes and discover that you don't have your glasses. You know that the following is true:

(a) If my glasses are at the kitchen table, then I saw them at breakfast.

(b) I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

(c) If I was reading the newspaper in living room, then my glasses are on the coffee table.

(d) I did not see my glasses at breakfast.

(e) If I was reading a book in the bed, then my glasses are on the nightstand.

(f) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

Where are the glasses?

Solution: We start by introducing variables for individual statements:

- p: My glasses are on the kitchen table.
- q: I saw my glasses at breakfast.
- r: I was reading the newspaper in the living room.
- s: I was reading the newspaper in the kitchen.
- t: My glasses are on the coffee table.
- u: I was reading a book in the bed.
- v: My glasses are on the nightstand.

Then, the premises of the argument become:

(a) $p \to q$
(b) $r \vee s$
(c) $r \to t$
(d) $\sim q$
(e) $u \to v$
(f) $s \to p$

Let's see what conclusion can be derived using the rules of inference we have learned so far.

	$p \to q$	by (a)
1.	$\sim q$	by (d)
$\therefore$	$\sim p$	by Modus Tollens

	$s \to p$	by (f)
2.	$\sim p$	by (1)
$\therefore$	$\sim s$	by Modus Tollens

	$r \vee s$	by (b)
3.	$\sim s$	by (2)
$\therefore$	r	by Elimination

	$r \rightarrow t$	by (c)
4.	r	by (3)
$\therefore$	t	by Modus Ponens

Therefore, t is true and the glasses are on the coffee table.

Fallacies

- Fallacy is an error in reasoning that leads to an invalid argument.
- Common errors:
  1. assuming what is to be proved without deriving it from the premises.
  2. jumping to a conclusion without using any of the valid inference rules
  3. converse error
  4. inverse error

Converse Error

Converse error results from an invalid argument form

$$\begin{array}{rl} & p \rightarrow q \\ & q \\ \therefore & p \end{array}$$

This argument is erroneous since it would be true if the converse statement

$$q \rightarrow p$$

was true instead of the original premise $p \rightarrow q$.

Inverse Error

Inverse error results from an invalid argument form

$$\begin{array}{rl} & p \to q \\ & \sim p \\ \therefore & \sim q \end{array}$$

This argument is erroneous since it would be true if the inverse statement

$$\sim p \to \sim q$$

was true instead of the original premise $p \to q$.

Contradictions and Valid Arguments

Contradiction Rule: If one can show that the assumption that p is false leads logically to a contradiction, then one can conclude that p is true.

- We are going to prove that the argument

  $$\begin{array}{rl} & \sim p \to \mathbf{c} \\ \therefore & p \end{array}$$

  is valid.

p	$\sim p$	$\mathbf{c}$	$\sim p \to \mathbf{c}$	p
T	F	F	T	T
F	T	F	F	F

Example

There is an island inhabited by two types of people: knights who always tell the truth and knaves who always lie. You visit the island and are approached by two natives who make the following statements:

> A says: B is a knight.
>
> B says: A and I are of opposite type.

What are A and B?

Solution: We will show that both A and B are knaves, using the contradiction rule.

Suppose A is a knight.	
$\therefore$ What A says is true.	(by definition)
$\therefore$ B is a also a knight.	(A's statement, which is true)
$\therefore$ What B says is true.	(by definition)
$\therefore$ A and B are of opposite type.	(B's statement, which is true)

This is a contradiction, since we derived that A and B are of opposite type, yet they are both knights.

$\therefore$ A is not a knight (Contradiction Rule)

$\therefore$ A is a knave (elimination; A is either a knight or a knave)
$\therefore$ What A says is false.
$\therefore$ B is not a knight.
$\therefore$ B is also a knave. (elimination)

Black Boxes and Gates

- Black box: implementation of a circuit.
- A black box describes the relation between input and output signals.
- The operation of a black box is completely specified by giving an input/output table.

Example

An example of an input/output table

An Input/Output Table

Input			Output
P	***Q***	***R***	***S***
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

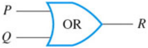
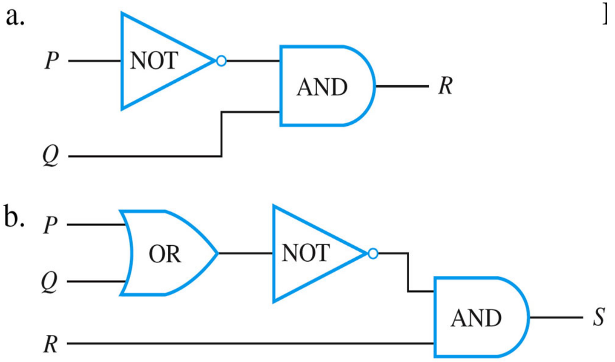
<table>
<tr><th>Type of Gate</th><th>Symbolic Representation</th><th>Action</th></tr>
<tr><td>NOT</td><td>
</td><td>
<table>
<tr><th>Input</th><th>Output</th></tr>
<tr><th>P</th><th>R</th></tr>
<tr><td>1</td><td>0</td></tr>
<tr><td>0</td><td>1</td></tr>
</table>
</td></tr>
<tr><td>AND</td><td>
</td><td>
<table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th>P</th><th>Q</th><th>R</th></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>1</td><td>0</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
</table>
</td></tr>
<tr><td>OR</td><td>
</td><td>
<table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th>P</th><th>Q</th><th>R</th></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
</table>
</td></tr>
</table>

Figure: Gates

Example

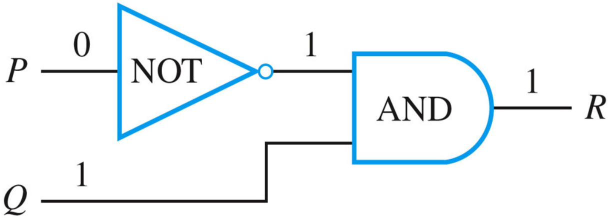
Indicate the output of the circuits given below for the given input signals.



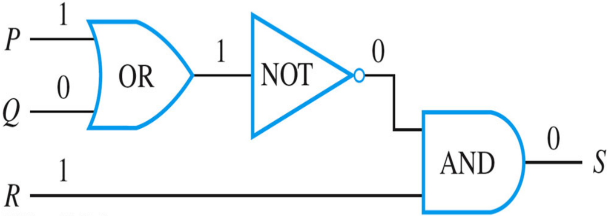
(a) Input signals: $P = 0$ and $Q = 1$

(b) Input signals: $P = 1$, $Q = 0$, $R = 1$.

Solution: (a)

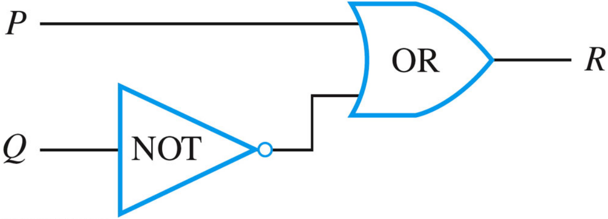


(b)



Example

Construct the input/output table for the following circuit

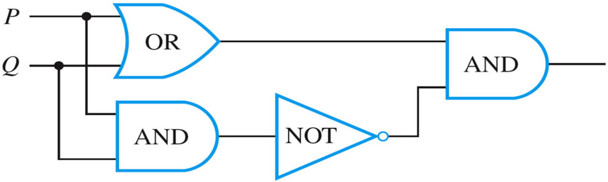


Solution:

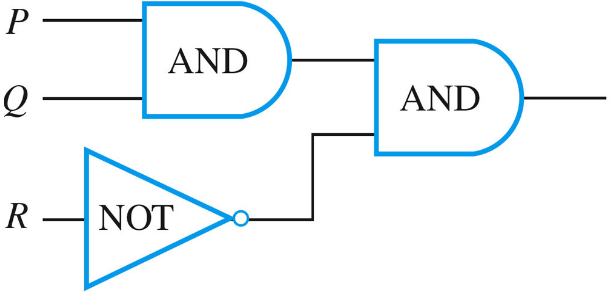
Input		Output
P	*Q*	*R*
1	1	1
1	0	1
0	1	0
0	0	1

Example

Find the Boolean expression that corresponds to the following two circuits. A dot indicates that two wires have been soldered; wires that cross without a dot are assumed not to touch.

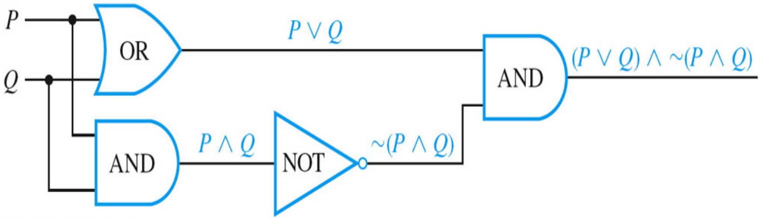


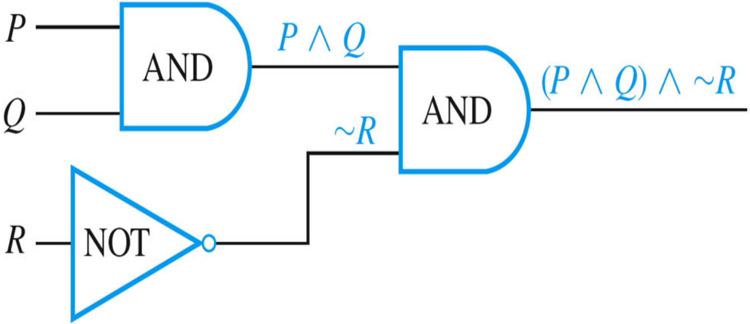
(a)



(b)

Solution:


P
Q
OR
AND
$P \vee Q$
$P \wedge Q$
NOT
$\sim(P \wedge Q)$
AND
$(P \vee Q) \wedge \sim(P \wedge Q)$


P
Q
R
AND
AND
NOT
P ∧ Q
~R
(P ∧ Q) ∧ ~R

- The input/output table for the circuit (b) from the previous example has the following form:

Input/Output Table for a Recognizer

P	Q	R	$(P \wedge Q) \wedge \sim R$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

- We notice that this circuit "recognizes" one particular combination of inputs:

$$P = 1, \quad Q = 1, \quad R = 0$$

Definition

A recognizer is a circuit that outputs 1 for exactly one combination of input signals and outputs 0's for all other input combinations.

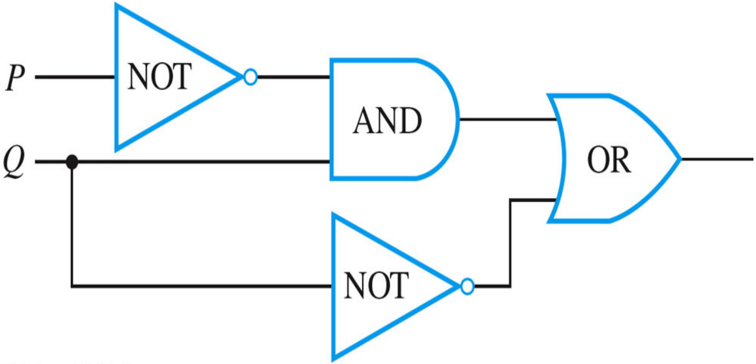
Example

Construct circuits for the following Boolean expression:

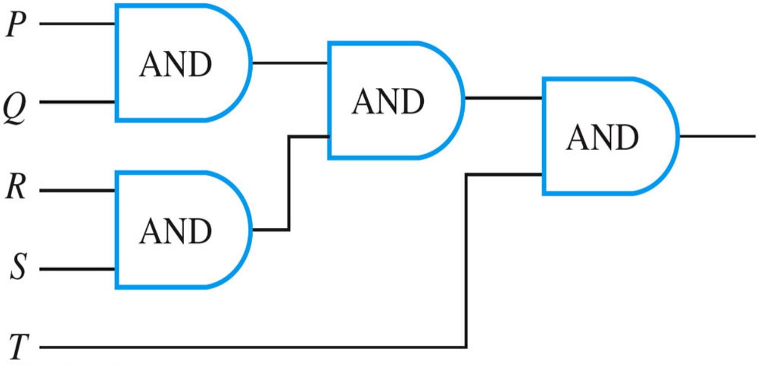
(a) $(\sim P \wedge Q) \vee \sim Q$

(b) $((P \wedge Q) \wedge (R \wedge S)) \wedge T$

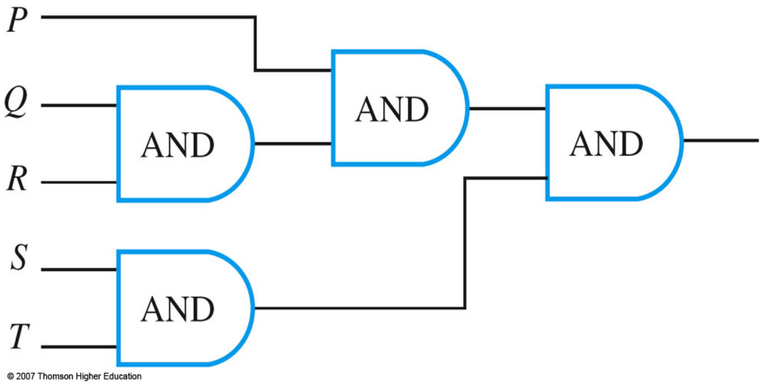
Solution: (a)



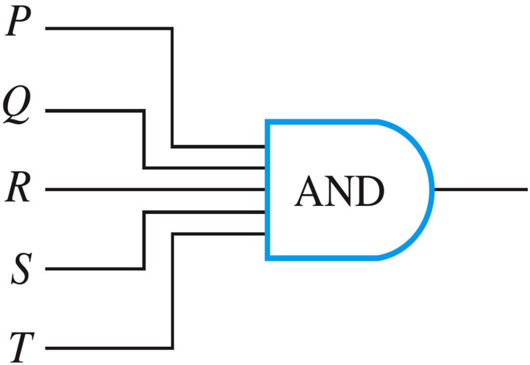
(b)



Because of the associativity of ∧, that circuit has the same input/output table as



In fact, we can view this circuit as a multiple-input AND-gate:



Example

Design a logical circuit for the following input/output table:

Input			Output
P	***Q***	***R***	***S***
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

Solution: We first identify the only possibilities for which the output signal will be 1:

(a) $P = Q = R = 1;$

(b) $P = 1,\ Q = 0,\ R = 1;$

(c) $P = 1, Q = R = 0$

Each such row will correspond to a simple Boolean expression:

(a) $P \wedge Q \wedge R$

(b) $P \wedge \sim Q \wedge R$

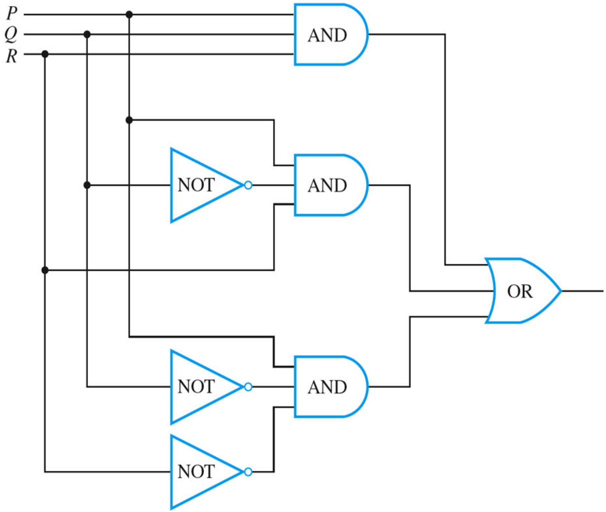
(c) $P \wedge \sim Q \wedge \sim R$

So, in order to produce a Boolean expression whose truth table is the input/output table we take the disjunction of these three possible combinations of P, Q and R which yield output 1:

$$(P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge R) \vee (P \wedge \sim Q \wedge \sim R)$$

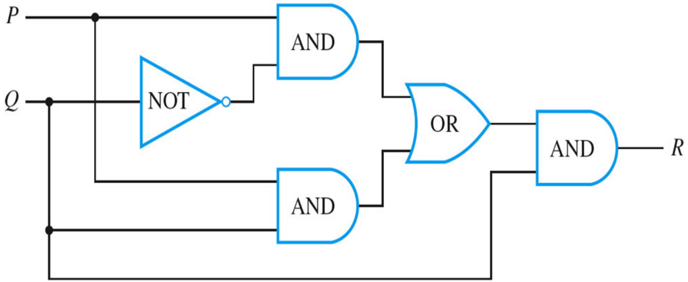
Remark

A Boolean expression of this type, i.e. a Boolean expression which is a disjunction of conjunctions of variables and their negations is called a disjunctive normal form.

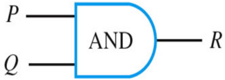

P
Q
R
AND
NOT
AND
OR
NOT
AND
NOT

Example

Consider the following two logical circuits:



(a)



(b)

It is relatively easy to see that both circuits have the same input/output table:

Input		Output
P	***Q***	***R***
1	1	1
1	0	0
0	1	0
0	0	0

Definition

We say that two logical circuits are equivalent if, and only if, their input/output tables are identical.

- To prove that two circuits are equivalent, one can first find their Boolean expressions and then prove that these expressions are equivalent as propositional formulas.

Example

In the previous example, the expression corresponding to circuit (a) is

$$((P \wedge \sim Q) \vee (P \wedge Q)) \wedge Q$$

It is a simple exercise to show that

$$((P \wedge \sim Q) \vee (P \wedge Q)) \wedge Q \equiv P \wedge Q$$

using the basic equivalences of propositional logic.