1302A – Notes – by Eric Hua

Contents

1.1 Systems of Linear Equations

Definition 1 *A* **linear equation** *in variables* $x_1, x_2, \dots, x_n$ *has the form*

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where the numbers $a_1, \dots, a_n \in \mathbb{R}$ *are the equation's coefficients and* $d \in \mathbb{R}$ *is the constant. An n-tuple* $(s_1, s_2, \dots, s_n) \in \mathbb{R}^n$ *is a* **solution** *of, or satisfies, that equation if substituting the numbers* s_1, ..., s_n *for the variables gives a true statement:* $a_1s_1 + a_2s_2 + \ldots + a_ns_n = d$.

A **system of linear equations**

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m \end{array}$$

has the solution $(s_1, s_2, \dots, s_n)$ *if that n-tuple is a solution of all of the equations in the system.*

Finding the set of all solutions is solving the system.

Example 1 *The ordered pair* $(-1, 5)$ *is a solution of this system.*

$$\begin{array}{ccccc} 3x_1 & + & 2x_2 & = & 7 \\ -x_1 & + & x_2 & = & 6 \end{array}$$

In contrast, (5,-1) is not a solution.

Definition 2 *If we have two linear systems and they have the same solution set then the two linear systems are called* **equivalent**.

Theorem 1 *The linear system has,*

1. *no solution*
2. *one solution*
3. *infinitely many solutions.*

In case 1, the linear system is called inconsistent. In case 2 or 3, the linear system is called consistent.

Example 2 *The system*

$$\begin{array}{rcrcl} 3x_1 & + & 2x_2 & = & 7 \\ -x_1 & + & x_2 & = & 6 \end{array}$$

has only one solution (-1,5).

Example 3 *The system*

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 7 \\ -2x_1 & - & 4x_2 & = & -14 \end{array}$$

has infinite solutions (7-2k, k).

Example 4 *The system*

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 7 \\ -2x_1 & - & hx_2 & = & k \end{array}$$

has no solution when $h = 4$ *and* $k \neq -14$*; one solution when* $h \neq 4$*; infinite solutions when* $h = 4$ *and* $k = -14$*.*

Matrices

Definition 3 *An* $m \times n$ *(*m *by* n*)* matrix A *with* m *rows and* n *columns with entries in* $\mathbb{R}$ *is a rectangular array of the form*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where $\forall (i, j) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}, \quad a_{ij} \in \mathbb{R}$*.*

As a shortcut, we often use the notation $A = [a_{ij}]$ to denote the matrix A with entries a_{ij}. Notice that when we refer to the matrix we put parentheses—as in "$[a_{ij}]$," and when we refer to a specific entry we do not use the surrounding parentheses—as in "a_{ij}."

Example 5

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

is a 2×3 *matrix and*

$$B = \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$$

is a 3×2 *matrix.*

To solving linear systems, we put all the coefficients of each variable aligned in columns to get the **coefficient matrix**. By adding an additional column to the coefficient matrix consisting of the values on the right hand side of the equal sign to give the **augmented matrix**.

Example 6 *Consider this linear system*

$$\begin{array}{rcrcrcl} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

$$\text{coefficient matrix} = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 5 & -2 \\ 1/3 & 2 & 0 \end{bmatrix}, \quad \text{augmented matrix} = \begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ 1/3 & 2 & 0 & 3 \end{bmatrix}.$$

Elementary row operations: There are three types of elementary row operations.

1. Replacement: Replace one row by the sum of itself and the multiple of another.
2. Interchange: Interchange two rows.
3. Scaling: Multiply all entries in a row by a non zero constant.

Definition 4 *Two matrices are row equivalent if one matrix can be transformed into another matrix by a sequence of elementary row operations.*

1.2 Row reduction and echelon forms

A **leading entry of a row**: is the leftmost, nonzero entry in the row (nonzero row).
Definition 5 *A rectangular matrix is in* **echelon form** *(row echelon form) if it has the following three properties.*

1. All non zero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zero.

Definition 6 *Given a matrix in echelon form, if it satisfies the following two conditions then it is in reduced echelon form (reduced row echelon form)*

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

Example 7 *Classify row echelon form, reduced row echelon form, or not in echelon form:*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, E = \begin{bmatrix} 2 & 3 & 4 & 0 & 1 \\ 0 & 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, G = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We are interested in performing row operations until one of these two matrix structures arises.

Uniqueness of the Reduced Echelon Form: Each matrix is row equivalent to one and only one reduced echelon matrix.

Example 8 *Look at the following matrix:*

$$\begin{bmatrix} 1 & h-1 & 1 & 0 & 1 \\ 0 & h & 1 & 0 & 1 \\ 0 & 0 & h-1 & 1 & 0 \end{bmatrix}$$

1) Find h such that the matrix is in reduced row echelon form; ($h = 1$)

2) Find h such that the matrix is in echelon form but not in reduced row echelon form; ($h \neq 1$ and $h \neq 0$)

3) Find h such that the matrix is not in row echelon form. ($h = 0$.)

Example 9 *Carry the following matrix to (1) Echelon form; (2) reduced echelon form:*
$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & 1 \\ 3 & -7 & 8 & -5 & 8 & 2 \\ 3 & -9 & 12 & -9 & 6 & 6 \end{bmatrix}.$$

Definition 7 *A pivot position in a matrix A is a location in A that corresponds to a leading entry in an echelon form of A. A pivot column is a column of A that contains a pivot position.*

Theorem 2 *(Existence and Uniqueness Theorem) A linear system is consistent* $\Leftrightarrow$ *the rightmost column of the augmented matrix is not a pivot column.*

Properties:

- The augmented matrix of a consistent linear system is row equivalent to a matrix with the last non-zero row
$$\begin{bmatrix} \cdots & * & * \end{bmatrix}.$$
- The augmented matrix of a inconsistent linear system is row equivalent to a matrix with the last non-zero row
$$\begin{bmatrix} 0 & \cdots & 0 & * \end{bmatrix}.$$

Example 10 *Solve the following system by using elementary row operations:*

$$\begin{array}{rcrcrcr} x & + & y & - & 2z & = & -2 \\ & & y & + & 3z & = & 7 \\ x & & & - & z & = & -1 \end{array}$$

We can start by going to echelon form:

$$\text{augmented matrix} = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 1 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 4 & 8 \end{bmatrix} \xrightarrow{\frac{1}{4}R_3} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 + 2R_3, R_2 - 3R_3} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

The answer is $x = 1$, $y = 1$, and $z = 2$.

Example 11 *Solve the following system by using elementary row operations:*

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 4 \\ & & 2x_2 & + & 2x_3 & = & 8 \end{array}$$

Solution: The general solution is

$$\begin{array}{rcl} x_1 & = & -7 - x_3 \\ x_2 & = & 4 - x_3 \\ x_3 & = & free \end{array}$$

Remark: x_1, x_2 here are basic variables, x_3 is called a free variable.

Remark: For a consistent system, if it has free variables, it has infinite solutions; if no free variable, then only one solution.

1.3 Vector Equations

Definition 8 *A matrix with one column is called a column vector and a matrix with one row a row vector.*

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, x_1 \in \mathbb{R}, \cdots x_n \in \mathbb{R} \right\}.$$

A vector in $\mathbb{R}^n$ is denoted by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Example 12

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [\vec{a}_1 \dots \vec{a_n}],$$

where $\vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \dots, \vec{a_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m$ *are called column vectors.*

Definition 9 *Vector addition:*

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n),$$

Scalar multiplication:

$$\lambda(a_1, a_2, \ldots, a_n) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n).$$

Properties: For $\vec{u}$, $\vec{v}$, $\vec{w} \in \mathbb{R}^m$ and scalars c, d,

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. $\vec{u} + 0 = \vec{u} = 0 + \vec{u}$
4. $\vec{u} + (-\vec{u}) = -\vec{u} + \vec{u} = 0$ and $-\vec{u} = (-1)\vec{u}$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
7. $c(d\vec{u}) = (cd)\vec{u}$
8. $1\vec{u} = \vec{u}$

Given a set of vectors $\vec{v}_1, \cdots, \vec{v}_m$ in $\mathbb{R}^n$ and scalars $x_1, \ldots, x_m$, if

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{y},$$

then $\vec{y}$ is called a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$.

Definition 10

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{b}$$

is called vector equation.

Sometimes we may be given a set of vectors and we want to express one as a linear combination of the others. In this situation our object is to find the coefficients that allow us to do this.

Example 13 *Let*

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -4 \\ -2 \\ -10 \end{bmatrix}.$$

Can $\vec{y}$ be a linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$?

Solution: $\vec{y}$ is a linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$ $\Leftrightarrow$ the vector equation

$$x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = \vec{y}$$

is consistent.

The augmented matrix of the vector equation:

$$\text{augmented matrix } = \begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 3 & 2 & -2 \\ 3 & 4 & -5 & -10 \end{bmatrix} \xrightarrow{R_2 - 2R_1, R_3 - 3R_1} \begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 4 & 6 \\ 0 & -2 & -2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 + 2R_2, R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 7 & 8 \\ 0 & -1 & 4 & 6 \\ 0 & 0 & -10 & -10 \end{bmatrix} \xrightarrow{\frac{1}{-10}R_3} \begin{bmatrix} 1 & 0 & 7 & 8 \\ 0 & -1 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 - 7R_3, R_2 - 4R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

$$\Rightarrow \vec{y} = \vec{u} - 2\vec{v} + \vec{w}.$$

Definition 11 *Given a set of vectors $\vec{v}_1, \cdots, \vec{v}_m$ in $\mathbb{R}^n$,*

$$Span\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m; \quad \text{where } c_1, c_2, ..., c_m \text{ are scalars}\}.$$

Example 14 *Are* $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *in* $span\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$?

Example 15 $span\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\} = \mathbb{R}^2$?

Linear Combinations in Applications

Example 16 *Consider an open economy with three products: coal-mining operation, electricity-generating plant, and automobile. To produce \$1 of coal, the mining operation must purchase \$0.5 of its own production, \$0.2 of electricity and \$0.1 worth of automobile for its transportation. To produce \$1 of electricity, it must purchase \$0.3 of its own production, \$0.4 of electricity and \$0.05 worth of automobile for its transportation.*

i) $\vec{b} = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.1 \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.05 \end{bmatrix}$ represents the cost per dollar of income for each product.

ii) $x\vec{b} + y\vec{c}$ represents the total cost of manufacturing x dollars of coal and y dollars of electricity.

1.4 Equation $A\vec{x} = \vec{b}$

Definition 12 *Let* $A = [\vec{a_1} \dots \vec{a_n}]$, $\quad \vec{a_1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \dots, \vec{a_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}, \in \mathbb{R}^m, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. *Define*

$$A\vec{x} = x_1\vec{a}_1 + \dots + x_n\vec{a}_n,$$

which is the linear combination of the columns of A.

$$A\vec{x} = \vec{b}$$

is called matrix equation.

Example 17 *The linear system :*

$$\begin{aligned} x_1 - x_4 &= 1 \\ x_1 - 2x_3 &= 2 \\ x_1 + 2x_2 + 3x_3 &= 3 \\ x_2 + 5x_3 &= 4 \end{aligned}$$

$$\Leftrightarrow$$

The vector equation:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 3 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\Leftrightarrow$$

The matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Theorem 3 *Let A be an $m \times n$ matrix. Then the following statements are equivalent, (either all true or all false).*

1. *For each $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.*
2. *Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.*
3. *The columns of A span $\mathbb{R}^m$.*
4. *A has a pivot position in every row.*

Properties: Some properties of matrix and vector multiplication are, if A is an $m \times n$ matrix, $\vec{u}, \vec{v} \in \mathbb{R}^n$, and c is a scalar, then,

1. $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
2. $A(c\vec{u}) = c(A\vec{u})$

Example 18 *Let* $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. *Describe $\vec{b}$ for which $A\vec{x} = \vec{b}$ has a solution.*

Sol:

$$[A|\vec{b}] = \begin{bmatrix} 1 & -3 & -4 & x \\ -3 & 2 & 6 & y \\ 5 & -1 & -8 & z \end{bmatrix} \xrightarrow{R_2 + 3R_1, R_3 - 5R_1} \begin{bmatrix} 1 & -3 & -4 & x \\ 0 & -7 & -6 & y + 3x \\ 0 & 14 & 12 & z - 5x \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & -3 & -4 & x \\ 0 & -7 & -6 & y + 3x \\ 0 & 0 & 0 & x + 2y + z \end{bmatrix}.$$

Thus, $A\vec{x} = \vec{b}$ has a solution $\Leftrightarrow x + 2y + z = 0$.

Example 19 *Let*

$$\vec{u} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix}.$$

Is $Span\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3$?

Sol: Yes, since $A = [\vec{u}\ \vec{v}\ \vec{w}]$ has pivot in each row.

1.5 Solution sets of linear systems

1. Homogeneous System

Definition 13 *A linear system $A\vec{x} = \vec{b}$ is called homogeneous if $\vec{b} = \vec{0}$. Otherwise, it is non-homogeneous. Let $\vec{x}$ be a solution of $A\vec{x} = \vec{0}$. If $\vec{x} = \vec{0}$, then $\vec{x}$ is called a trivial solution; if $\vec{x} \neq \vec{0}$, then $\vec{x}$ is called a non-trivial solution.*

Example 20 *Consider the following system of linear equations*

$$\begin{aligned} x_1 + 4x_2 - 8x_3 &= 0 \\ 2x_1 + 5x_2 - 7x_3 &= 0 \\ -3x_1 - 7x_2 + kx_3 &= 0 \end{aligned}$$

(i) Find value(s) of k such that the system has only trivial solution.
(ii) Find value(s) of k such that the system has non-trivial solutions.
(iii) For the value(s) of k in (ii), describe the solution set.

Solution: (i)

$$\text{augmented matrix} = \begin{bmatrix} 1 & 4 & -8 & 0 \\ 2 & 5 & -7 & 0 \\ -3 & -7 & k & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1, R_3 + 3R_1} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & k-24 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 5 & k-24 & 0 \end{bmatrix} \xrightarrow{R_3 - 5R_2} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & k-9 & 0 \end{bmatrix}$$

Hence, for $k \neq 9$, the system has only trivial solution.

(ii) For $k = 9$, the system has non-trivial solution.

(iii) When $k = 9$, from the discussion in (i), we have

$$\text{augmented matrix} = \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 4R_2} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{aligned} x_1 + 4x_3 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}$$

i.e.,

$$\begin{aligned} x_1 &= -4t \\ x_2 &= 3t \\ x_3 &= t\ (free) \end{aligned}$$

Then the general solution is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4t \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}.$$

2. Parametric Vector Form

Whenever a solution set is described explicitly with vectors, we say that the solution is in parametric vector form.

Example 21 *Consider the following system of linear equations*

$$\begin{aligned} x_1 - 2x_2 - 9x_3 + 5x_4 &= 0 \\ x_2 + 2x_3 - 6x_4 &= 0. \end{aligned}$$

Describe the solution set in parametric vector form.

Solution:

$$\text{augmented matrix} = \begin{bmatrix} 1 & -2 & -9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & -5 & -7 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \Rightarrow$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 + 7x_4 \\ -2x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 6x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}.$$

3. Non-homogeneous System

Theorem 4 *If $\vec{p}$ is a solution of $A\vec{x}=\vec{b}$, and $\vec{v_h}$ is a solution of $A\vec{x}=\vec{0}$, then $\vec{p}+\vec{v_h}$ is a solution of $A\vec{x}=\vec{b}$*

Example 22 *Let* $A=\begin{bmatrix}1 & -3 & -4\\ -3 & 8 & 6\\ 5 & -13 & -8\end{bmatrix}, \vec{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \vec{b}=\begin{bmatrix}1\\ -2\\ 3\end{bmatrix}$. *Describe all solutions of $A\vec{x}=\vec{b}$.*

Sol:

$$[A|\vec{b}]=\begin{bmatrix}1 & -3 & -4 & 1\\ -3 & 8 & 6 & -2\\ 5 & -13 & -8 & 3\end{bmatrix}\xrightarrow{R_2+3R_1, R_3-5R_1}\begin{bmatrix}1 & -3 & -4 & 1\\ 0 & -1 & -6 & 1\\ 0 & 2 & 12 & -2\end{bmatrix}$$

$$\xrightarrow{R_3+2R_2}\begin{bmatrix}1 & -3 & -4 & 1\\ 0 & -1 & -6 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}\xrightarrow{R_1-3R_2}\begin{bmatrix}1 & 0 & 14 & 4\\ 0 & -1 & -6 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}\xrightarrow{-R_2}\begin{bmatrix}1 & 0 & 14 & 4\\ 0 & 1 & 6 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Thus,

$$\vec{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}4-14x_3\\ 1-6x_3\\ x_3\end{bmatrix}=\begin{bmatrix}4\\ 1\\ 0\end{bmatrix}+\begin{bmatrix}-14x_3\\ -6x_3\\ x_3\end{bmatrix}=\begin{bmatrix}4\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}-14\\ -6\\ 1\end{bmatrix}.$$

Example 23 *Given the augmented matrix* $\begin{bmatrix}1 & 5 & 0 & 0 & -2\\ 0 & 0 & 1 & 0 & 12\\ 0 & 0 & 0 & 1 & -3\end{bmatrix}$.

(i) Is it consistent? If yes, write the solution in parametric vector form.

(ii) Solve the corresponding homogeneous system.

1.6 Applications of Linear Systems

1. Homogeneous System in economics

Equilibrium price or market clearing price: The price such that the amount of goods or services sought by buyers is equal to the amount of goods or services produced by sellers.

Example 24 *Consider an economy with three industries: coal-mining operation, electricity-generating plant and an auto-manufacturing plant.*

The distribution of the output from coal: 20% to electricity, 80% to auto;

The distribution of the output from electricity: 50% to Coal, 10% to electricity itself, 40% to auto;

The distribution of the output from auto: 10% to Coal, 30% to electricity, 60% to auto.

(i) Construct the input-output table (a sector looks down a column to see where its output goes, and it looks across a row to see what it needs as inputs).

(ii) Find the equilibrium prices.

Solution. (i)

Distribution of output from:			
coal	electricity	auto	Purchased by
0	0.5	0.1	coal
0.2	0.1	0.3	electricity
0.8	0.4	0.6	electricity

(ii) Let p_c, p_e and p_a be the dollar values of outputs of coal-mining, electricity, auto-manufacturing, respectively. Let $\mathbf{p} = \begin{bmatrix} p_c \\ p_e \\ p_a \end{bmatrix}$ be the price vector. Then

$$\begin{aligned} p_c &= 0.5p_e + 0.1p_a \\ p_e &= 0.2p_c + 0.1p_e + 0.3p_a \\ p_a &= 0.8p_c + 0.4p_e + 0.6p_a, \end{aligned}$$

which gives

$$\begin{aligned} p_c - 0.5p_e - 0.1p_a &= 0 \\ -0.2p_c + 0.9p_e - 0.3p_a &= 0 \\ -0.8p_c - 0.4p_e + 0.4p_a &= 0, \end{aligned}$$

The augmented matrix is

$$[A|\vec{b}] = \begin{bmatrix} 1 & -0.5 & -0.1 & 0 \\ -0.2 & 0.9 & -0.3 & 0 \\ -0.8 & -0.4 & 0.4 & 0 \end{bmatrix} \xrightarrow{R_2+0.2R_1, R_3+0.8R_1} \begin{bmatrix} 1 & -0.5 & -0.1 & 0 \\ 0 & 0.8 & -0.32 & 0 \\ 0 & -0.8 & 0.32 & 0 \end{bmatrix}$$

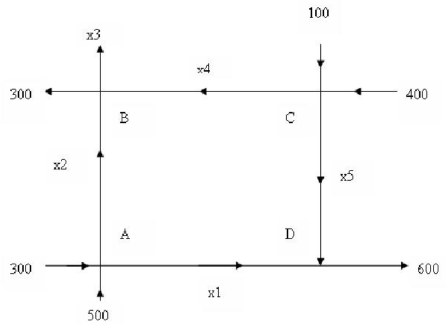
$$\xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -0.5 & -0.1 & 0 \\ 0 & 0.8 & -0.32 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{1/0.8R_2} \begin{bmatrix} 1 & -0.5 & -0.1 & 0 \\ 0 & 1 & -0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+0.5R_2} \begin{bmatrix} 1 & 0 & -0.3 & 0 \\ 0 & 1 & -0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$\vec{p} = p_a \begin{bmatrix} 0.3 \\ 0.4 \\ 1 \end{bmatrix}.$$

2. Network Flow

Example: The network below shows the traffic flow (vehicles/hour) over several one-way streets in downtown Baltimore. Determine the general flow pattern for the network. Find maximum x_1.



Solution: Introducing variables in the graph. Set "flow in" = "flow out":

Intersection	Flow in		Flow out
A	300+500	=	$x_1 + x_2$
B	$x_2 + x_4$	=	$300 + x_3$
C	100+400	=	$x_4 + x_5$
D	$x_1 + x_5$	=	600
Total	1300	=	$900 + x_3$

Therefore we have the following linear system:

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & & & & & & & = & 800 \\ & & x_2 & - & x_3 & + & x_4 & & & = & 300 \\ & & & & & & x_4 & + & x_5 & = & 500 \\ x_1 & + & & & & & & & x_5 & = & 600 \\ & & x_3 & & & & & & & = & 400 \end{array}$$

The augmented matrix is:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{bmatrix} \xrightarrow{R_4 - R_1} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & -1 & 0 & 0 & 1 & -200 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2, R_4 + R_2} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 500 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & -1 & 1 & 1 & 100 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{bmatrix} \xrightarrow{R_1 + R_4, R_2 - R_4} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & -1 & 1 & 1 & 100 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & -1 & 0 & 0 & -400 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & -1 & 0 & 0 & -400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{bmatrix}$$

$$\xrightarrow{(-1)R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{bmatrix} \xrightarrow{R_5 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We get the general solution:

$$\begin{aligned} x_1 &= 600 - x_5 \\ x_2 &= 200 + x_5 \\ x_3 &= 400 \\ x_4 &= 500 - x_5 \\ x_5 &= free. \end{aligned}$$

Maximum x_1=600, where when $x_5 = 0$.

2.1 Matrix operations

- Diagonal matrix: Except for entries on diagonal (main diagonal), all other entries are 0,
- Zero matrix: all entries are 0,
- Identity matrix I_n: all entries on diagonal are 1, other entries are 0.

Scalar multiplication and addition of matrices: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices, r be a number. Then

$$rA = [ra_{ij}], \quad A + B = [a_{ij} + b_{ij}].$$

Remark. You can only add matrices of the same size. Also, two matrices are equal if they are the same size and corresponding entries are equal.

Example 25

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 0+4 & -1+5 & 1+6 \\ 1+7 & 2+8 & 3+9 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 7 \\ 8 & 10 & 12 \end{bmatrix}.$$

$$3 \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & 6 \\ 0 & 9 \end{bmatrix}.$$

Properties:

Let A, B, C be matrices of the same size and let r and s be scalars.

1. A+B = B+A
2. (A+B)+C = A+(B+C)
3. A+0 = A
4. r(A+B) = rA + rB
5. (r+s)A = rA + sA
6. r(sA) = (rs)A

Matrix multiplication: Let $A = [a_{ij}]_{m\times r}$ and $B = [b_{ij}]_{r\times n}$. Then

$$AB = [c_{ij}]_{m\times n},$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}.$$

Example 26

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 1(7)+2(9)+3(11) & 1(8)+2(10)+3(12) \\ 4(7)+5(9)+6(11) & 4(8)+5(10)+6(12) \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Remark. In order to have the product AB of two matrices A and B, the number of columns of A must equal the number of rows of B. So, if A is an $m \times r$ and B is an $s \times n$ matrix, in order to have the product AB, we need r = s. The resulting matrix AB will be an $m \times n$ matrix.

Properties of matrix multiplication: Let A, B, C be matrices for which sums and products are defined.

1. A(BC) = (AB)C (associativity)
2. A(B+C)= AB + AC (Left distributivity)
3. (B+C)A = BA + CA (Right distributivity)
4. r(AB) = (rA)B = A(rB)
5. $I_mA = A = AI_n$, here A is m x n.
6. In general, $AB \neq BA$.

Transpose of a matrix: Given a matrix A then transpose of A is a matrix denoted by A^T, whose rows are the columns of A and whose columns are the rows of A.

Example 27

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties of Transpose:

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$ where r is a scalar.
4. $(AB)^T = B^TA^T$.

2.2 The Inverse of a Matrix

Definition 14 *Given an $n \times n$ matrix A, the inverse of A is an $n \times n$ matrix B such that*

$$BA = AB = I,$$

where I is the $n \times n$ identity. The inverse of A is denoted by A^{-1}.

Example 28 *The inverse of 2×2 matrix: If $ad - bc \neq 0$, then*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Theorem 5 *If A is an invertible $n \times n$ matrix, then for each $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.*

Properties of inverses:

1. If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. If A and B are invertible then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. If A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Elementary Matrices:

An elementary matrix is a matrix obtained by performing one elementary row operation onto an identity matrix. Every elementary matrix is invertible. The inverse of an elementary matrix E is again an elementary matrix E^{-1} and represents the elementary row operation that transforms E into the identity matrix.

Theorem 6 *An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n and in this case, any sequence of elementary row operations that reduce A to I_n also transforms I_n into A^{-1}.*

Example 29 *Find A^{-1}, where*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{bmatrix}.$$

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 6 & 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1, R_3 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 5 & -2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -2 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right].$$

Thus

$$A^{-1} = \begin{bmatrix} 14 & -2 & -3 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

2.3 Characterization of Invertible Matrices

Theorem 7 *(The Invertible Matrix Theorem) Let A be a square $n \times n$ matrix. Then the following statements are equivalent.*

1. *A is an invertible matrix.*
2. *A is row equivalent to the identity matrix.*
3. *A has n pivot positions.*
4. *The equation $A\vec{x} = \vec{0}$ has only the trivial solution.*
5. *The columns of A form a linearly independent set.*
6. *The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is 1:1.*
7. *The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b} \in \mathbb{R}^n$.*
8. *The columns of A span $\mathbb{R}^n$.*
9. *The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is onto.*
10. *There is an $n \times n$ matrix C such that $CA = I$.*
11. *There is an $n \times n$ matrix D such that $AD = I$.*
12. *A^T is invertible.*

Remark. X and Y are equivalent means X implies Y and Y implies X.

Remark. Another name for an invertible matrix is non singular and similarly another name for a non-invertible matrix is singular.

Invertible Linear Transformation:

Definition 15 *Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, T is invertible if there is another transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $S(T\vec{x})) = T(S(\vec{x})) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. S is called the inverse of T, $S = T^{-1}$, and vice versa.*

Theorem 8 *Let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with a standard matrix A. T_A is invertible if and only if A is invertible. The inverse transformation of T_A is T_A^{-1} with standard matrix A^{-1}.*

Example 30 *Let* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *be given by* $T(x_1, x_2, x_3) = (x_1 - 2x_2 + 3x_3, x_2 + 3x_3, x_3)$. *Find* $S = T^{-1}$.

Sol:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \Rightarrow A^{-1} = \begin{bmatrix} 1 & 2 & -9 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}, \Rightarrow$$

$$T^{-1}(x_1, x_2, x_3) = A^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -9 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1 + 2x_2 - 9x_3, x_2 - 3x_3, x_3).$$

2.6 The Leontief Input-output Model

The Leontief Input-output Model is an economic model measuring how changes in one sector affect other sectors.

Suppose a nation's economy is divided into n sectors that produce goods or services.

- $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots x_n \end{bmatrix}$: A production vector in $\mathbb{R}^n$ that lists the units produced by each sector (output).
- $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots d_n \end{bmatrix}$: A final demand vector that lists the values of goods and services demanded from the various sectors by the nonproductive part of economy.
  - The model is closed if $\mathbf{d} = \mathbf{0}$.
  - The model is closed if $\mathbf{d} \neq \mathbf{0}$.
- C: Consumption matrix (input-output matrix). Each column of C lists units consumed by all sectors from producing 1 unit of each sector. For example, the first column of C lists units consumed by all sectors from producing 1 unit of sector 1.
- $C\mathbf{x}$: Intermediate demand created by the producers themselves for goods they need as inputs for their own production.

The Leontief Input-output Model: $C\mathbf{x} + \mathbf{d} = \mathbf{x}$.

Theorem 9 *If C and $\mathbf{d}$ have nonnegative entries and if each column sum of C is less than 1, then $(I-C)^{-1}$ exists, and the production vector $\mathbf{x} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d}$ has nonnegative entries and is the unique solution of the Leontief Input-output Model.*

Example 31 *Consider an open economy with three industries: coal-mining operation, electricity-generating plant and an auto-manufacturing plant. To produce \$1 of coal, the mining operation must purchase \$0.5 of its own production, \$0.2 of electricity and \$1 worth of automobile for its transportation. To produce \$1 of electricity, it takes \$0.8 of electricity and \$0.4 of automobile. Finally, to produce \$1 worth of automobile, the auto-manufacturing plant must purchase \$0.25 of coal, \$0.1 of electricity. Assume also that during a period of one week, the economy has an exterior demand of \$100 worth of coal, \$500 worth of electricity, and \$700 worth of automobile. Find the production level of each of the three industries in that period of one week in order to exactly satisfy both the internal and the external demands.*

Solution. Let x_1, x_2 and x_3 be the dollar values of outputs of coal-mining, electricity, auto-manufacturing, respectively. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the production vector.

The input-output matrix of this economy and the demand vector are

$$C = \begin{bmatrix} 0.5 & 0 & 0.25 \\ 0.2 & 0.8 & 0.1 \\ 1 & 0.4 & 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 100 \\ 500 \\ 700 \end{bmatrix}.$$

Then

$$I - C = \begin{bmatrix} 0.5 & 0 & -0.25 \\ -0.2 & 0.2 & -0.1 \\ -1 & -0.4 & 1 \end{bmatrix}, \quad (I-C)^{-1} = \begin{bmatrix} 16 & 10 & 5 \\ 30 & 25 & 10 \\ 28 & 20 & 10 \end{bmatrix},$$

which gives

$$\mathbf{x} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d} == \begin{bmatrix} 10100 \\ 22500 \\ 19800 \end{bmatrix}.$$

So, the total output of the coal-mining operation must be \$10100, the total output for the electricity-generating plant is \$22500 and the total output for the auto-manufacturing plant is \$19800.

Example 32 *Consider an open economy with three industries: coal-mining operation, electricity-generating plant and an auto-manufacturing plant. To produce \$1 of coal, the mining operation must purchase \$0.1 of its own production, \$0.30 of electricity and \$0.1 worth of* *automobile for its transportation. To produce \$1 of electricity, it takes \$0.25 of coal, \$0.4 of electricity and \$0.15 of automobile. Finally, to produce \$1 worth of automobile, the auto-manufacturing plant must purchase \$0.2 of coal, \$0.5 of electricity and consume \$0.1 of automobile. Assume also that during a period of one week, the economy has an exterior demand of \$50,000 worth of coal, \$75,000 worth of electricity, and \$125,000 worth of autos. Find the production level of each of the three industries in that period of one week in order to exactly satisfy both the internal and the external demands.*

Solution. The input-output matrix of this economy and the demand vector are

$$C = \begin{bmatrix} 0.1 & 0.25 & 0.2 \\ 0.3 & 0.4 & 0.5 \\ 0.1 & 0.15 & 0.1 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 50000 \\ 75000 \\ 125000 \end{bmatrix}.$$

Then

$$I - C = \begin{bmatrix} 0.9 & -0.25 & -0.2 \\ -0.3 & 0.6 & -0.5 \\ -0.1 & -0.15 & 0.9 \end{bmatrix}, \quad (I - C)^{-1} = \begin{bmatrix} 1.464 & 0.803 & 0.771 \\ 1.007 & 2.488 & 1.606 \\ 0.330 & 0.503 & 1.464 \end{bmatrix},$$

which gives

$$\mathbf{x} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d} = \begin{bmatrix} 1.464 & 0.803 & 0.771 \\ 1.007 & 2.488 & 1.606 \\ 0.330 & 0.503 & 1.464 \end{bmatrix} \begin{bmatrix} 50000 \\ 75000 \\ 125000 \end{bmatrix} = \begin{bmatrix} 229790 \\ 437795 \\ 237402 \end{bmatrix}.$$

So, the total output of the coal-mining operation must be \$229790, the total output for the electricity-generating plant is \$437795 and the total output for the auto-manufacturing plant is \$237402.

Appendix B: Complex Numbers

Complex number is

$$z = a + bi, \quad a \in \mathbb{R}, b \in \mathbb{R}, i^2 = -1.$$

Addition and multiplication:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i.$$

The conjugate of $z = a+bi$ is: $\bar{z} = a-bi$. We have

$$z\bar{z} = a^2+b^2.$$

The modulus of z is defined by

$$|z| = \sqrt{a^2+b^2}.$$

Properties

1. $\bar{z} = z$ if and only if z is real,

2. $\overline{z+w} = \bar{z}+\bar{w}$,

3. $\overline{zw} = \bar{z}\bar{w}$,

4. $|wz| = |w||z|$,

5. $|z+w| \leq |z|+|w|$ (triangle inequality),

6. let $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$ (De Moivre's Theorem).

Example 33 *Simplify* $\frac{2+3i}{4-5i}$.

Solution:

$$\frac{2+3i}{4-5i} = \frac{(2+3i)(4+5i)}{(4-5i)(4+5i)} = \frac{8+10i+12i-15}{4^2+5^2} = \frac{-7}{41} + i\frac{22}{41}.$$

Example 34 *Let* $z = \sqrt{3}+i$. *Calculate* z^6.

Sol: We write z as:

$$z = 2(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2e^{i\pi/6}.$$

$$z^6 = 2^6 e^{(i\pi/6)6} = 2^6 e^{i\pi} = 64(\cos\pi + i\sin\pi) = -64.$$

Example 35 *Solve the following equation* $2x^2-4x+4=0$.

3.1 Introduction to Determinants

Definition 16 *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *The determinant of* A *is defined as*

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For a $n \times n$ *matrix* A, *let* A_{ij} *be the matrix obtained from* A *by deleting the i-th row and j-th column. The* $(i,j)^{th}$ *cofactor of* A *is the number,*

$$c_{ij} = (-1)^{i+j} \det A_{ij}.$$

$$\det A = a_{i1}c_{i1} + a_{i2}c_{i2} + ... + a_{in}c_{in},$$

which is called a cofactor expansion across the i-th row. Similarly,

$$\det A = a_{1j}c_{1j} + a_{2j}c_{2j} + ... + a_{nj}c_{nj},$$

which is called a cofactor expansion across the j-th column.

Example 36 *Calculate* $\det A$, *where*

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}.$$

Solution: We do cofactor expansion across the 2nd row.

$$\begin{aligned} \det A &= a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23} \\ &= 2(-1)^{2+1} \det \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} + (-1)^{2+2} \det \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} + (-1)^{2+3} \det \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \\ &= 2(14) + (-13) + 5 = 20. \end{aligned}$$

Definition 17 *A triangular matrix is a matrix that is all zeros either above or below the diagonal. An upper triangular matrix means all entries below the main diagonal are zero; an lower triangular matrix means all entries above the main diagonal are zero.*

Theorem 10 *If* A *is a triangular matrix then* $detA$ *is the product of the entries on the main diagonal of* A.

Example 37 *Calculate* $\det A$*, where*

$$A = \begin{bmatrix} 5 & 3 & 5 & 7 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 12 \end{bmatrix}.$$

Solution: A is an upper triangular matrix. $\det A = 5(1)(2)(12) = 120$.

Theorem 11 *If* A *is a* 3×3 *matrix,* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, *then*

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

Remark. This comes from the three main diagonals and three other diagonals by repeating the first two columns.

3.2 Properties of Determinants

Theorem 12 *(Row Operations)*

1. If a multiple of one row of A *is added to another row to produce a matrix* B *then* $\det B = \det A$.

2. If two rows of A *are interchanged to produce* B *then detB = -detA.*

3. If one row of A *is multiplied by* k *to produce* B *then det* $B = k$ *det* A.

Example 38 *Let* $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$. *Find* $\det(A)$.

Solution:

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 - 3R_3} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

If we denote the above three new matrices as B, C, D, then

$$\det A = \det B = \det C = -\det D = -(1)(-2)(-1) = -2.$$

Properties of determinants:

1. If A is a square matrix then $\det A^T = \det A$.
2. $\det(AB) = \det A \det B$.
3. If A is an $n \times n$ matrix and c a scalar then $\det(cA) = c^n \det A$.
4. If A is invertible then $det(A^{-1}) = \frac{1}{\det A}$.
5. A square matrix A is invertible $\Leftrightarrow \det A \neq 0$.

Example 39 *Let A, B and C be 3×3 invertible matrices, $det(A) = 3$, $det(B) = 5$, $det(C) = 6$. Calculate $det(A^{-1}C(-2B))$.*

Solution.

$$det(A^{-1}C(-2B)) = det(A^{-1})det(C)(-2)^3 det(B) = -8\frac{1}{det(A)}det(C)det(B)$$

$$= -8\frac{1}{3}(6)5 = -80.$$

5.1 Eigenvectors and Eigenvalues

Definition 18 *An eigenvector of an $n \times n$ matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. $\vec{x}$ is called the eigenvector corresponding to λ.*

To determine whether a given value λ is an eigenvalue of a matrix A we need to find a non-zero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. This is the same as determining whether the matrix equation

$$(A - \lambda I)\vec{x} = 0$$

has a non-trivial solution.

Example 40 *Let* $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$,$\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. *Note that*

$$A\vec{u} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7\vec{u}, \quad A\vec{v} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\vec{v}, \quad A\vec{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \neq \lambda\vec{w}.$$

Thus $\vec{u}$ is an eigenvector corresponding to $\lambda = 7$, $\vec{v}$ is an eigenvector corresponding to $\lambda = -4$, $\vec{w}$ is not an eigenvector.

Definition 19 *The set of all eigenvectors for a particular eigenvalue λ of a matrix A is a subspace and so is called the eigenspace of A corresponding to λ.*

Example 41 *The eigenvalues of a triangular matrix are the entries on its main diagonal.*

Theorem 13 *If $\vec{v_1}, ..., \vec{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots,$ λ_r of an $n \times n$ matrix A, then the set $\{\vec{v_1}, ..., \vec{v_r}\}$ is linearly independent.*

5.2 The Characteristic Equation

Characteristic equation: $\det(A - \lambda I)$ is called the characteristic polynomial of A and

$$\det(A - \lambda I) = 0$$

is called the characteristic equation.

Theorem 14 *The solutions of the characteristic equation are the eigenvalues of A.*

Example 42 *Find all eigenvalues of A, where*

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & -1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2(1-\lambda).$$

The solutions of the characteristic equation $\det(A - \lambda I) = 0$ are 3,3,1.

Algebraic and Geometric multiplicity:

- The algebraic multiplicity of an eigenvalue is equal to the number of times it is a root of the characteristic equation.
- The geometric multiplicity of an eigenvalue is the dimension of its eigenspace.

Example 43 *Let*

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

The eigenvalue 3 has algebraic multiplicity 2 and the eigenvalue 1 has algebraic multiplicity 1. Find the corresponding eigenspaces and their geometric multiplicities.

Solution: When $\lambda = 3$,

$$A - \lambda I = A - 3I = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - 3I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenspace has a basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. The geometric multiplicity of the eigenvalue 3 is 1.

When $\lambda = 1$,

$$A - \lambda I = A - I = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - \frac{1}{2}R_2} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The eigenspace has a basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$. The geometric multiplicity of the eigenvalue 1 is 1.

The Invertible Matrix Theorem (continued): Let A be an $n \times n$ matrix. Then the following statements are equivalent to, A is an invertible matrix."

19. The number 0 is not an eigenvalue of A.
20. The determinant of A is not zero.

Similar matrices: Two matrices A and B are similar if there is an invertible matrix P such that,

$$A = PBP^{-1}.$$

Theorem 15 *If $n \times n$ matrices A and B are similar, then they have the same characteristics polynomial and hence the same eigenvalues (with the same multiplicities).*

Proof.

$$\begin{aligned}
\det(A - \lambda I) &= \det(PBP^{-1} - \lambda PP^{-1}) \\
&= \det[P(B - \lambda I)P^{-1}) \\
&= \det(P)\det(B - \lambda I)\det(P^{-1}) \\
&= \det(P)\det(B - \lambda I)\frac{1}{\det(P)} \\
&= \det(B - \lambda I).
\end{aligned}$$

Application to Dynamical System:

Example 44 *Let* $A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$, $\vec{x_0} = \begin{bmatrix} 0.1 \\ 0.7 \end{bmatrix}$, $\vec{x_{k+1}} = A\vec{x_k}$ $(k = 0, 1, ...,)$. *Analyze the long-term behavior of the system.*

Solution: a) Find all eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 0.95 - \lambda & 0.03 \\ 0.05 & 0.97 - \lambda \end{vmatrix} = \lambda^2 - 1.92\lambda + 0.92 = 0, \rightarrow$$

$\lambda = 1, 0.92$

b) Find basis to each eigenspace:

When $\lambda = 1$,

$$A - \lambda I = A - I = \begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} -0.05 & 0.03 \\ 0 & 0 \end{bmatrix}.$$

Thus $(A-I)\vec{x}=0$ has the solution

$$\vec{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=t\begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

The eigenspace has a basis $\left\{\vec{V_1}=\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right\}$.

When $\lambda=0.92$,

$$A-\lambda I=A-0.92I=\begin{bmatrix} 0.03 & 0.03 \\ 0.05 & 0.05 \end{bmatrix}\xrightarrow{R_2-\frac{5}{3}R_1}\begin{bmatrix} 0.03 & 0.03 \\ 0 & 0 \end{bmatrix}.$$

Thus $(A-0.93I)\vec{x}=0$ has the solution

$$\vec{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=t\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The eigenspace has a basis $\left\{\vec{V_2}=\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

c) Express $\vec{x_0}=\begin{bmatrix} 0.1 \\ 0.7 \end{bmatrix}$ as a linear combination of $\vec{V_1}$ and $\vec{V_2}$:

$$\vec{x_0}=0.1\vec{V_1}+0.2\vec{V_2},\Rightarrow$$

$$\vec{x_1}=A\vec{x_0}=0.1A\vec{V_1}+0.2A\vec{V_2}=0.1(1)\vec{V_1}+0.2(0.92)\vec{V_2}.$$

In general

$$\vec{x_k}=0.1(1)^k\vec{V_1}+0.2(0.92)^k\vec{V_2}\to 0.1\vec{V_1}$$

as $k\to\infty$.

5.3 Diagonalization

A diagonal matrix is a matrix with non zero values along its diagonal and zeros on its off diagonal entries. Well only be considering diagonal matrices that are square.

Definition 20 *If A is a square $n\times n$ matrix and A is similar to a diagonal matrix D then A is said to be diagonalizable.*

Theorem 16 *(Diagonalization Theorem) Let A be an $n\times n$ matrix.*

- *A is diagonalizable if and only if A has n linearly independent eigenvectors. If $A = PDP^{-1}$, where D is a diagonal matrix, the the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.*

- *If A has n distinct eigenvalues, then A is diagonalizable.*

- *A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n, if and only if the dimension of the eigenspace for each eigenvalue equals the algebraic multiplicity of the eigenvalue. (Generally, the dimension of the eigenspace for each eigenvalue is less than or equal to the algebraic multiplicity of the eigenvalue).*

For an $n \times n$ matrix A, if A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to the eigenvalue λ_k, $k = 1, ..., p$, then the total collection of vectors in the sets $\mathcal{B}_1$,..., $\mathcal{B}_p$ forms an eigenvector basis of $\mathbb{R}^n$.

Example 45 $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ *is diagonalizable: 4 distinct eigenvalues.*

$B = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ *is not diagonalizable: $\lambda = 4$, one eigenvector* $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Example 46 *Let* $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

1) Find P and D such that $A = PDP^{-1}$.
2) Calculate A^4.

Sol: 1) $\det(A - \lambda I) = (\lambda - 5)(\lambda + 2)$.
When $\lambda = 5$: $\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;
When $\lambda = -2$: $\vec{x} = \frac{x_2}{4} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Thus

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}; \text{ or } P = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}, D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}.$$

2) Let $P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$, then $P^{-1} = \frac{1}{7}\begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$.

$$A^4 = \{PDP^{-1}\}^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}^4 \frac{1}{7} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 625 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 364 & 261 \\ 348 & 277 \end{bmatrix}.$$

4.9 Applications to Markov Chains

- Probability vector: A vector with nonnegative entries that add up to 1.
- Stochastic matrix: A square matrix whose columns are probability vectors.
- Markov chain: A sequence of probability vectors $x_0, x_1,$ together with a stochastic matrix P such that $x_{k+1} = Px_k, k = 0, 1, 2,$

Example 47 *Suppose a car rental agency has three locations in Ottawa: Downtown location (labeled A), East end location (labeled B) and a West end location (labeled C). The agency has a group of delivery drivers to serve all three locations. The agency's statistician has determined the following:*

1. At the Downtown location, 30% are delivered in Downtown area, 30% are delivered in the East end, and 40% are delivered in the West end

2. At the East end location, 40% are delivered in Downtown area, 40% are delivered in the East end, and 20% are delivered in the West end

3. At the West end location, 50% are delivered in Downtown area, 30% are delivered in the East end, and 20% are delivered in the West end.

Find the probabilities of being at each location after 2 deliveries. If $x_0 = [100]^T$, what is x_2?

Solution. We model this problem with the following matrix:

$$P = \begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}.$$

P is called the transition matrix (a stochastic matrix) of the above system.

$$P^2 = \begin{bmatrix} 0.41 & 0.38 & 0.37 \\ 0.33 & 0.34 & 0.33 \\ 0.26 & 0.28 & 0.30 \end{bmatrix}.$$

Remark. The elements on each column still add to 1 and each element is between 0 and 1.

Steady-State Vectors

If P is a stochastic matrix, then a steady-state vector for P is a probability vector $\vec{q}$ such that

$$P\vec{q} = \vec{q}, \Rightarrow (P - I)\vec{q} = \vec{0}.$$

Example 48 *Let*

$$P = \begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}.$$

Find a steady-state vector for P.

Solution:

$$[P - I|\vec{0}] = \begin{bmatrix} -0.7 & 0.4 & 0.5 & | & 0 \\ 0.3 & -0.6 & 0.3 & | & 0 \\ 0.4 & 0.2 & -0.8 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & -7 & | & 0 \\ 0 & 5 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

By solving (P-I)x=0 we have: $x1 = 7/5x3, x2 = 6/5x3$. So $\vec{q} = \begin{bmatrix} 7/18 \\ 6/18 \\ 5/18 \end{bmatrix}$.

Convergence

A stochastic matrix P is called regular if, for some integer k, all entries of P^k are strictly positive.

Theorem. If P is a regular stochastic matrix, then P has a unique steady-state vector $\vec{q}$. Further, Markov chain $\{x_k\}$ converges to $\vec{q}$ as $k \to \infty$.

Example. $P = \begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}$ is regular. $\vec{q} = \begin{bmatrix} 7/18 \\ 6/18 \\ 5/18 \end{bmatrix}$. This shows that after a large number of deliveries, it no longer matters which location we were in when we started, we have (approximately) a 33.3% Chance of being at location B.

1.7 Linear Independence

Definition 21 *A set of vectors* $\{\vec{v}_1, \cdots, \vec{v}_m\}$ *in* $\mathbb{R}^n$ *is linearly independent if the vector equation*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{0}$$

has only the trivial solution. The set is said to be linearly dependent if there is a non-trivial solution to the vector equation.

Theorem 17 *The columns of* A *are linearly independent if and only if* $A\vec{x} = \vec{0}$ *ONLY has the trivial solution.*

Example 49 $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$ *is linearly independent,* $\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ *is linearly dependent.*

Example 50 *Given* $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. *Show that* $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ *is linearly dependent and find the linear combination.*

Solution: $-2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$.

Theorem 18

1. *A set of two vectors is linearly dependent if and only if one of the vectors is a multiple of the other.*
2. *A set of two or more vectors is linearly dependent if and only if at least one vector may be written as a linearly combination of the others.*
3. *If a set contains more vectors than entries in each vector, then the set is linearly dependent.*
4. *If the zero vector is in a set of vectors, then the set of vectors is linearly dependent.*

2.8 Subspaces of $\mathbb{R}^n$

Definition 22 *A subspace is any subset V of $\mathbb{R}^n$ that satisfies,*

1. $\vec{0} \in V$
2. *For each* $\vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} \in V$ *(Closed under addition).*
3. *For each* $\vec{x} \in V$ *and scalar* c, $c\vec{x} \in V$ *(Closed under scalar multiplication.)*

Example 51

1. $S = \{\vec{0}\}$ *is a subspace.*
2. *Given a set of vectors* $\vec{v}_1, \cdots, \vec{v}_m$ *in* $\mathbb{R}^n$,

$$Span\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m; \quad where\ c_1, c_2, ..., c_m\ are\ scalars\}$$

   is a subspace.

3. $H = \left\{ \begin{bmatrix} x \\ x \\ 3x - 2y \end{bmatrix} | x, y \in \mathbb{R} \right\}$ *is a subspace of* $\mathbb{R}^3$;
4. $H = \{(a, a+1, a-1) | a \in \mathbb{R}\}$ *is not a subspace of* $\mathbb{R}^3$;
5. $H = \{(a-1, b, c) | a, b, c \in \mathbb{R}\}$ *is a subspace of* $\mathbb{R}^3$.

Definition 23 *Let* $A = [\vec{a}_1 \ldots \vec{a_n}]_{m\times n}$. *The columns of A span a subspace of $\mathbb{R}^n$ called the column space of A and is denoted by ColA:*

$$ColA = Span\{\vec{a}_1, \ \ldots, \vec{a_n}\}.$$

The null space of a matrix A is the set of all solutions to the homogeneous equation $A\vec{x} = \vec{0}$ and is a subspace of $\mathbb{R}^n$. The null space of A is denoted by NulA.

$$NulA = \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}.$$

Definition 24 *A basis for a subspace H is a linearly independent set of vectors that spans H. We denote it by $\mathcal{B}_{\mathcal{H}}$. When H is clear, we just write the basis as $\mathcal{B}$. The number of vectors in a basis for a subspace H is called the dimension of H and is denoted by* $\dim H$.

Properties of basis:

- There is more than one basis for a subspace.

- A basis is the largest spanning set of linearly independent vectors for a subspace.
- Pivot columns can be a basis for ColA.

Example 52 *Let*

$$A = [\vec{a}_1\, \vec{a}_2\, \vec{a}_3\, \vec{a}_4\, \vec{a}_5] = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find $\mathcal{B}_{ColA}$ and $\mathcal{B}_{NulA}$.

Sol: Pivot columns are $\vec{a}_1\, \vec{a}_4$. So $\mathcal{B}_{ColA} = \{\vec{a}_1, \vec{a}_4\}$.

To find $\mathcal{B}_{NulA}$, consider $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \vec{0}$.

$$\textit{Augmented matrix} = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 & -17 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 - 2x_3 + 17x_5 \\ x_2 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 17 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

Therefore,

$$\mathcal{B}_{NulA} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 17 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

4.1 Vector Spaces and Subspaces

Definition. Let V be a set of elements. If the operations addition and scalar multiplication are defined in V satisfying the following axioms:

A1. If $\vec{u}, \vec{v} \in V$, then $\vec{u} + \vec{v} \in V$.

A2. If $\vec{u}, \vec{v} \in V$, then $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

A3. If $\vec{u}, \vec{v}, \vec{w} \in V$, then $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.

A4. There exists an element, denoted by $\vec{0}$, in V, such that $\vec{0} + \vec{u} = \vec{u}$ for every $\vec{u}$.

A5. For every $\vec{u} \in V$, there exists an element, denoted by $-\vec{u}$, in V such that $\vec{u}+(-\vec{u}) = \vec{0}$.

S1. Operation scalar multiplication is defined for every number c and every $\vec{u}$ in V, and $c\vec{u} \in V$.

S2. Operation scalar multiplication satisfies the distributive law: $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$.

S3. Operation scalar multiplication satisfies the second distributive law: $(c + d)\vec{u} = c\vec{u} + d\vec{u}$.

S4. Operation scalar multiplication satisfies the associative law: $(cd)\vec{u} = c(d\vec{u})$.

S5. For every element $u \in V$, $1\vec{u} = \vec{u}$.

Then V is called a vector space.

Example.

(i) $\mathbb{R}^n$ is a vector space.

Properties.

(i) $\vec{w} + \vec{v} = \vec{u} + \vec{v}$ implies $\vec{w} = \vec{u}$.

(ii) $0\vec{v} = \vec{0}$.

(iii) $c\vec{0} = \vec{0}$.

(iv) av = 0 implies a = 0 or v = 0.

(v) (1)v = v.

(vi) (a)v = (av) = a(v).

Subspaces.

A set U is a subspace of a vector space V if U is a vector space with respect to the operations of V. U is a subspace of V if

(i) the zero vector is in U;

(ii) if x is in U, then ax is in U for any scalar a, and

(iii) if x, y are in U, then x + y is in U.

Examples

(i) $\{\vec{0}\}$ is a subspace. A subspace that is not $\{\vec{0}\}$ is a proper subspace.

(ii) A line through the origin in the space is a subspace; A plane through the origin in the space is a subspace.

(iii) $S_1 = \{(s, 2s, 3)|s \in R\}$ is not a subspace.

(iv) $S_2 = \{\begin{bmatrix} s \\ s^2 \end{bmatrix} |s \in R\}$ is not a subspace. It does not satisfy (ii).

(v) $S_3 = \{(s, t)|s^2 = t^2, s, t \in R\}$ is not a subspace. It does not satisfy (iii).

(vi) $S_4 = \{\begin{bmatrix} s \\ 2s + 3t \\ 5t \end{bmatrix} |s, t \in R\}$ is a subspace.

(vii) $\mathbb{R}^n$ is a subspace of itself.

(viii) $S_5 = \{\begin{bmatrix} s + 1 \\ t \end{bmatrix} |s, t \in R\}$ is a subspace.

(ix) $S_6 = \{\begin{bmatrix} a \\ b \\ c \end{bmatrix} |a = 3b + 2c, a, b, c \in R\}$ is a subspace.

Spanning sets as a subspace

Let $S = \{v_1, v_2, , v_k\}$. span $S = \{c_1v_1 + c_2v_2 + ... + c_kv_k|c_1, , c_k \in \mathbb{R}\}$. It is a subspace of V, called the span of S, denoted by span S.

Example. The span of a single non-zero vector in the space is a line through the origin.

Let A be a subspace, and let S be a subset of A. If span S = A, then S is a spanning set of A. In particular, A itself is a spanning set of A. A subspace generally has more than one spanning set.

Properties

(i) If $X \in S$, then $X \in spanS$.

(ii) If a subspace W contains every vector in S, then W contains span S.

As an example of using the second property, $span\{X + Y, X, Y\} = span\{X, Y\}$.

(iii) If $\vec{b}$ is a linear combination of $v_1, v_2, ..., v_k$, then

span$S = \{\vec{b}, v_1, v_2, , v_k\} =$ span$S = \{v_1, v_2, ..., v_k\}$.

(iv) $R^n = span\{E_1, E_2, ..., E_n\}$.

(v) null A = the span of the basic solutions of AX=0.

(vi) im A = the span of the columns of A.

Example 53 *(i) Verify that [1 2 0 1]^T is in span{[2 1 2 0]^T , [0 -3 2 2]^T }.*

Solution: The corresponding system is consistent.

(ii) Verify that the set of vectors $S = \{[1\ 2\ 3]^T, [-1\ 0\ 1]^T, [2\ 1\ -1]^T\}$ *spans* R^3.

Solution: For any $[a\ b\ c]^T$ *in* R^3, *The corresponding system is consistent.*

(iii) Find a,b such that $X = [a\ b\ a+b\ a-b]^T$ *is in span*$\{X1, X2, X3\}$, *where* $X1 = [1\ 1\ 1\ 1]^T$, $X2 = [1\ 0\ 1\ 2]^T$, $X3 = [-1\ 0\ 1\ 0]^T$.

2.9 Dimension and Rank

Definition 25 *Given a basis* $\mathcal{B} = \{\vec{v}_1, ..., \vec{v}_p\}$ *for a subspace* H. *Let* $\vec{x} \in H$. *Then* $\vec{x}$ *may be written as a linear combination of* $\vec{v}_1, ..., \vec{v}_p$*:*

$$\vec{x} = c_1\vec{v}_1 + ... + c_p\vec{v}_p.$$

The weights $c_1, ..., c_p$ *are called the coordinates of* $\vec{x}$ *relative to the basis* $\mathcal{B}$. *These coordinates may be written as a vector*

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

called the coordinate vector of $\vec{x}$ *relative to* $\mathcal{B}$*, (or the* $\mathcal{B}$*-coordinate vector of* $\vec{x}$*).*

Example 54 *Let* $A = [\vec{a}_1\, \vec{a}_2\, \vec{a}_3\, \vec{a}_4\, \vec{a}_5] = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 4 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

1) Pivot columns are $\vec{a}_1\, \vec{a}_3$. *So* $\mathcal{B}_{ColA} = \{\vec{a}_1, \vec{a}_3\}$.

2)

$$[\vec{a}_2]_{\mathcal{B}_{ColA}} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, [\vec{a}_4]_{\mathcal{B}_{ColA}} = \begin{bmatrix} 3/2 \\ 7/4 \end{bmatrix}, [\vec{a}_5]_{\mathcal{B}_{ColA}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

3) $\dim \mathcal{B}_{ColA} = 2$.

Note: The order of the vectors in the basis $\mathcal{B}$ influences the coordinate vector $[\vec{x}]_{\mathcal{B}}$.

Definition 26 *The rank of a matrix* A *is equal to the number of pivot positions in* A. *This in turn equals the number of pivot columns of* A *which equals the number of vectors in the basis for* $ColA$. $rankA = \dim ColA$.

$rankA = \dim ColA$.

The Rank Theorem: If a matrix A has n columns then

$$dimColA + dimNulA = n$$

or

$$rankA + \dim NulA = n.$$

The Invertible Matrix Theorem (continued): Let A be an $n \times n$ matrix. Then the following statements are equivalent to, A is an invertible matrix."

13. The columns of A form a basis for $\mathbb{R}^n$.
14. Col A = $\mathbb{R}^n$
15. dim ColA = n
16. rank A = n
17. Nul A = $\{0\}$
18. dim NulA = 0.

4.2 Null Spaces, Column Spaces, and Linear Transformations

(i) Null space and column space: Let $A = [\vec{a}_1 \dots \vec{a_n}]_{m\times n}$. The columns of A span a subspace of $\mathbb{R}^n$ called the column space of A and is denoted by ColA:

$$ColA = Span\{\vec{a}_1, \dots, \vec{a_n}\}.$$

The null space of a matrix A is the set of all solutions to the homogeneous equation $A\vec{x} = \vec{0}$ and is a subspace of $\mathbb{R}^n$. The null space of A is denoted by NulA.

$$NulA = \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}.$$

Example 55 *Given the matrix* $A = \begin{bmatrix} 1 & 5 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$.

(i) Find null A.

(ii) Find Col A.

Remark. The number of parameters = the number of solutions in a set of basic solutions = the number of variables rank of A.

(ii) Kernel and Range of a Linear Transformation

Let V and W be vector spaces. Then a linear transformation L from V to W is a function with domain V and range a subset of W satisfying

1) L(u + v) = L(u) + L(v)

2) L(cu) = cL(u)

for any vectors u and v in V and scalar c.

Kernel $L = \{v \in V | L(v) = 0\}$, Range $L = \{L(v) | v \in V\}$.

Example. Let V be the vector space of (infinitely) differentiable functions and define D to be the function from V to V given by $D(f(t)) = f'(t)$. Then D is a linear transformation since

$D(f(t) + g(t)) = (f(t) + g(t))' = f'(t) + g'(t) = D(f(t)) + D(g(t))$.

$D(cf(t)) = (cf(t))' = cf'(t) = cD(f(t))$.

Kernel $D = \{c | c \in R\}$.