1341C. W11 Test 2 solutions

1. Let A be an 11×6 matrix such that $Ax = 0$ has only the trivial solution $x = 0$.

- What is the rank of A?
- Do the columns of A span $\mathbf{R}^{11}$?

A. 0, Yes
B. 6, Yes
C. 6, No
D. 8, Yes
E. 8, No
F. 2, Yes

$\Rightarrow$ cols of A are l.i., so $\dim \operatorname{col} A = \operatorname{rank} A = 6$

Since $\dim \mathbb{R}^{11} = 11 < 6 = \dim \operatorname{col} A$, the columns of A cannot span $\mathbb{R}^{11}$

If A is 9×6 instead (and $Ax = 0 \Rightarrow x = 0$), then $\operatorname{rank} A = 6$ and the six column of A cannot span $\mathbb{R}^9$

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2. For which value of α does the vector $(2, 3, \alpha)$ belong to the subspace of $\mathbf{R}^3$ spanned by $(1, 0, 3)$ and $(3, 2, 1)$?

A. 0
B. 6
B. -6
D. 1
E. -1
F. 1/2

We find α s.t. $(A|b) = \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 2 & 3 \\ 3 & 1 & \alpha \end{array}\right]$ is consistent:

$$[A|b] \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 2 & 3 \\ 0 & -8 & \alpha - 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & \alpha + 6 \end{array}\right].$$

This represents a consistent system $\Leftrightarrow \alpha = -6$

OR: Find β s.t. $(A|b) = \left[\begin{array}{cc|c} 1 & 3 & 2 \\ -1 & 1 & 1 \\ -1 & 1 & \beta \end{array}\right]$ is consistent:

$$[A|b] \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 4 & 3 \\ 0 & 4 & \beta + 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & \beta - 1 \end{array}\right].$$

So $\beta = 1$ for the system to be consistent

3. If $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, then the second row of B^{-1} is:

A. $[1\ \ 0\ \ -1]$
B. $[-1\ \ 1\ \ 0]$
C. $[0\ \ 1\ \ -1]$
D. $[0\ \ -1\ \ 1]$
E. $[1\ \ -1\ \ 0]$
F. None of the above

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & \\ 0 & 1 & 1 & I_3 \\ 0 & 0 & 1 & \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & & * & \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

The second row will not change now, so C is correct.

If $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $[B \mid I_3] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} * & * & * & * & * & * \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right].$$

Hence, in this case, the second row is given in D.

4. A Norwegian Blue parrot is advised by a nutritionist to take 14 units of vitamin A, 15 units of vitamin D and 29 units of vitamin E each day. The parrot can choose from the three brands I, II and III, and the amount of each vitamin in each capsule of the various brands is given below:

	I	II	III
vitamin A	2	1	1
vitamin D	3	3	0
vitamin E	5	4	1

This parrot is not capable of taking fractions of a capsule.

a) After defining your variables, write down a system of equations in these variables, together with all constraints, that determine the possible combinations of the numbers of capsules of each brand that will provide exactly the required amounts of vitamins for the parrot. (Do not perform any operations on your equations: this is done for you in (b). *Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

Let x = # capsules of type I taken each day
y = " II "
z = " III "

Then $x, y, z \geq 0$, $x, y, z \in \mathbb{Z}$, and

$2x + y + z = 14$ (to satisfy vit A req.)
$3x + 3y = 15$ (" D ")
$5x + 4y + z = 29$ (" E ")

b) The reduced row-echelon form of the augmented matrix of the system in part (a) is:

$$\begin{array}{c} \quad\quad s \quad \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 9 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$x = 9 - s$
$y = -4 + s$; $s \in \mathbb{R}$
$z = s$

[7 B

(Question 4 continued)

4c) Find all possible combinations of the numbers of capsules of each brand that will provide exactly the required amounts of vitamins for the parrot.

Since $x \geq 0, \quad s \leq 9$

$y \geq 0 \quad s \geq 4 \quad ; \; x, y, z \in \mathbb{Z}$

$z \geq 0 \quad s \geq 0 \quad \Leftrightarrow s \in \mathbb{Z}.$

Hence the possible combinations are

$(x, y, z) = (9-s, s-4, s), \; 4 \leq s \leq 9,$

$s \in \mathbb{Z}.$

4d) If the respective costs (in cents) per capsule of brands I, II and II are 7, 3 and 2, determine the choice which will minimize the total cost each day, and give this minimum cost per day.

total cost $= 7x + 3y + 2z$

$= 7(9-s) + 3(s-4) + 2s = 51 - 2s.$

$(s \in \mathbb{Z}, \; 4 \leq s \leq 9)$

This is minimized when $s = 9$ and the total cost is 33 cents/day. Then $x = 0, y = 5, z = 9.$

If respective costs are 7, 3 & 5, the total cost is then

$7x + 3y + 5z = 7(9-s) + 3(s-4) + 5s = 51 + s, \; (s \in \mathbb{Z}, 4 \leq s \leq 9).$

This is minimized when $s = 4$, so $x = 5, y = 0, z = 4$, and the total cost is then 55 cents/day

5. Suppose $v_1 = (1,0,2,1)$, $v_2 = (1,1,3,2)$, $v_3 = (0,1,1,1)$, $v_4 = (1,2,4,3)$ and define

$$V = \text{span}\{v_1, v_2, v_3, v_4\},$$

and

$$A = [\, v_1 \quad v_2 \quad v_3 \quad v_4 \,]$$

(i.e. A is the 4×4 matrix whose jth **column** is v_j, $j = 1, \ldots, 4$).

a) Find a basis for V which is a **subset** of the given spanning set $\{v_1, v_2, v_3, v_4\}$.

b) Find a basis for $\ker A = \{x \in \mathbf{R}^4 \mid Ax = 0\}$, and hence find its dimension.

c) Find a basis for col A.

d) Extend your basis for col A from part (c) to a basis of $\mathbf{R}^4$.

a) We use the column space algorithm:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 1 & 4 \\ 1 & 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\sim \begin{array}{c} \begin{array}{cccc} & & s & t \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \text{for (b)} \right)$$

Hence $\{v_1, v_2\}$ is a basis for V, (and $\{v_1, v_2\} \subset \{v_1, v_2, v_3, v_4\}$)

b) From (a) the general soln to $Ax = 0$ is

$$(x_1, x_2, x_3, x_4) = (s+t, -s-2t, 0, t) = s(1,-1,1,0) + t(1,-2,0,1)$$

Hence, $\{\underbrace{(1,-1,1,0)}_{w_1}, \underbrace{(1,-2,0,1)}_{w_2}\}$ spans $\ker A$. Since $sw_1 + tw_2 = 0 \iff s = t = 0$ (look at $3^{rd}, 4^{th}$ components), $\{w_1, w_2\}$ is a basis for $\ker A$. Thus $\dim \ker A = 2$

c) This was done in (a)! Thus, $\{v_1, v_2\}$ is a basis for col A.

d) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 2 \\ & u_3 & & \\ & u_4 & & \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ & u_3 & & \\ & u_4 & & \end{bmatrix}$. So choose $u_3 = (0,0,1,0)$, $u_4 = (0,0,1,0)$

Then, rank $\begin{bmatrix} v_1 \\ v_2 \\ u_3 \\ u_4 \end{bmatrix} = 4$, so $\{v_1, v_2, u_3, u_4\}$ is a basis of $\mathbb{R}^4$.

6. State whether the following are true (always), or may be false, in the box after the statement. You must justify your answer: if true, explain why, if not, give an explicit example (with numbers!) to show it is false.

Suppose A is an $n \times n$ matrix such that is a vector $x \in \mathbf{R}^n$, with $x \neq 0$, and $Ax = 0$.

a) The matrix A is invertible. (You must justify your answer with an example or by using the definition and properties of the inverse: you cannot simply state a theorem or 'fact' from class.)

If $Ax = 0$ and A were invertible, $\boxed{\text{FALSE}}$

$x = (A^{-1}A)x = A^{-1}(Ax) = A^{-1}(0) = 0$, contradicting

$x \neq 0$.

e.g. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $Ax = 0$ but

A is not invertible! (Such an example is possible $\forall n$.)

b) The rank of A is n. (You must justify your answer with an example or by using what you know about linear systems: you cannot simply state a theorem or 'fact' from class.)

$\boxed{\text{FALSE}}$

e.g. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

rank $A = 0 < 2 = n$.

c) The columns of A are linearly dependent. (You must justify your answer with an example or by using the definition of linear dependence: you cannot simply state a theorem or 'fact' from class.)

Hint: Write $A = [c_1 \ldots c_n]$ *in block column form, write* $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, *and consider* Ax.

$\boxed{\text{TRUE}}$

Since $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \neq 0$, at least one of the $x_i \neq 0$, $1 \leq i \leq n$.

But $Ax = [c_1 \ldots c_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 c_1 + \cdots + x_n c_n = 0$,

with at least one $x_i \neq 0$ ($1 \leq i \leq n$). Hence, by def^n, $\{c_1, \ldots, c_n\}$ is linearly dependent.

7. Let A and B be two matrices such that AB is defined.

a) Prove carefully that rank $AB \leq$ rank A (*Hint: show that* $\dim col(AB) \leq \dim col(A)$.)

b) Prove carefully that rank $AB \leq$ rank B (*Hint: show that* $\dim \text{row}\,(AB) \leq \dim \text{row}\,(B)$.)

a) Write $B = [b_1 \ldots b_m]$. Then $AB = [Ab_1 \ldots Ab_m]$, so $col(AB) = \text{Span}\{Ab_1, \ldots, Ab_m\}$. But $\forall i$, $1 \le i \le m$, Ab_i is a l.c. of the columns of A, so $Ab_i \in col(A)$, $\forall i$, $1 \le i \le m$. Hence $col(AB) \subseteq col(A)$. Hence

$$\text{rank}(AB) = \dim col(AB) \le \dim col(A) = \text{rank } A,$$

i.e. $\text{rank } AB \le \text{rank } A$.

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b) Write $A = \begin{bmatrix} r_1 \\ \vdots \\ r_p \end{bmatrix}$. Then $AB = \begin{bmatrix} r_1 B \\ \vdots \\ r_p B \end{bmatrix}$

Hence $\text{row}(AB) = \text{Span}\{r_1 B, \ldots, r_p B\}$.

But if we write $B = \begin{bmatrix} s_1 \\ \vdots \\ s_k \end{bmatrix}$ and $r_i = [a_{i1} \ldots a_{ik}]$,

then $r_i B = [a_{i1} \ldots a_{ik}] \begin{bmatrix} s_1 \\ \vdots \\ s_k \end{bmatrix} = a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{ik}s_k$,

so $r_i B \in \text{row}(B)$, $\forall i$, $1 \le i \le p$. Hence $\text{row}(AB) \subseteq \text{row}(B)$.

Thus, $\text{rank } AB = \dim \text{row}(AB) \le \dim \text{row}(B) = \text{rank}(B)$

i.e. $\text{rank } AB \le \text{rank } B$. [See next page for an alternate proof]

7(b) Using (a), we know that

$$\mathrm{col}(B^tA^t) \subseteq \mathrm{col}(B^t) \quad *$$

But $\mathrm{col}(B^tA^t) = \mathrm{col}((AB)^t) = \mathrm{row}(AB)$

and $\mathrm{col}(B^t) = \mathrm{row}(B)$,

so we conclude from $*$ that

$\mathrm{row}(AB) \subseteq \mathrm{row}(B)$, as desired.